

Nonlinear theory of a “shear-current” effect and mean-field magnetic dynamos

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The nonlinear theory of a “shear-current” effect in a nonrotating and nonhelical homogeneous turbulence with an imposed mean velocity shear is developed. The shear-current effect is associated with the $\bar{\mathbf{W}} \times \bar{\mathbf{J}}$ term in the mean electromotive force and causes the generation of the mean magnetic field even in a nonrotating and nonhelical homogeneous turbulence (where $\bar{\mathbf{W}}$ is the mean vorticity and $\bar{\mathbf{J}}$ is the mean electric current). It is found that there is no quenching of the nonlinear shear-current effect contrary to the quenching of the nonlinear α effect, the nonlinear turbulent magnetic diffusion, etc. During the nonlinear growth of the mean magnetic field, the shear-current effect only changes its sign at some value $\bar{\mathbf{B}}_*$ of the mean magnetic field. The magnitude $\bar{\mathbf{B}}_*$ determines the level of the saturated mean magnetic field which is less than the equipartition field. It is shown that the background magnetic fluctuations due to the small-scale dynamo enhance the shear-current effect and reduce the magnitude $\bar{\mathbf{B}}_*$. When the level of the background magnetic fluctuations is larger than 1/3 of the kinetic energy of the turbulence, the mean magnetic field can be generated due to the shear-current effect for an arbitrary exponent of the energy spectrum of the velocity fluctuations.

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I. INTRODUCTION

The magnetic fields of the Sun, solar-type stars, galaxies, and planets are believed to be generated by a dynamo process (see, e.g., [1–10]). In the framework of the mean-field approach, the large-scale magnetic field $\bar{\mathbf{B}}$ is determined by the induction equation

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times [\bar{\mathbf{U}} \times \bar{\mathbf{B}} + \mathcal{E}(\bar{\mathbf{B}}) - \eta \nabla \times \bar{\mathbf{B}}], \quad (1)$$

where $\bar{\mathbf{U}}$ is the mean velocity and η is the magnetic diffusion due to the electrical conductivity of fluid. The mean electromotive force $\mathcal{E}(\bar{\mathbf{B}}) = \langle \mathbf{u} \times \mathbf{b} \rangle$ is given by

$$\begin{aligned} \mathcal{E}_i(\bar{\mathbf{B}}) = & \alpha_{ij}(\bar{\mathbf{B}})\bar{B}_j - \eta_{ij}(\bar{\mathbf{B}})(\nabla \times \bar{\mathbf{B}})_j + [\mathbf{V}^{\text{eff}}(\bar{\mathbf{B}}) \times \bar{\mathbf{B}}]_i \\ & - [\delta(\bar{\mathbf{B}})(\nabla \times \bar{\mathbf{B}})]_i - \kappa_{ijk}(\bar{\mathbf{B}})(\partial \hat{B})_{jk}, \end{aligned} \quad (2)$$

where \mathbf{u} and \mathbf{b} are fluctuations of the velocity and magnetic field, respectively, angular brackets denote ensemble averaging, $(\partial \hat{B})_{ij} = (\nabla_i \bar{B}_j + \nabla_j \bar{B}_i)/2$ is the symmetric part of the gradient tensor of the mean magnetic field $\nabla_i \bar{B}_j$ —i.e., $\nabla_i \bar{B}_j = (\partial \hat{B})_{ij} + \varepsilon_{ijn}(\nabla \times \bar{\mathbf{B}})_n/2$ —and ε_{ijk} is the Levi-Civita tensor. Here $\alpha_{ij}(\bar{\mathbf{B}})$ and $\eta_{ij}(\bar{\mathbf{B}})$ determine the α effect and turbulent magnetic diffusion, respectively, $\mathbf{V}^{\text{eff}}(\bar{\mathbf{B}})$ is the effective drift velocity of the magnetic field, $\kappa_{ijk}(\bar{\mathbf{B}})$ describes a contribution to the mean electromotive force related to the symmetric parts of the gradient tensor of the mean magnetic field, $(\partial \hat{B})_{ij}$, and arises in an anisotropic turbulence, and the $\delta(\bar{\mathbf{B}})$

term determines a nontrivial behavior of the mean magnetic field in an anisotropic turbulence.

The mean magnetic field can be generated in a helical rotating turbulence due to the α effect described by the $\alpha_{ij}(\bar{\mathbf{B}})\bar{B}_j$ term in the mean electromotive force. When the rotation is nonuniform, the generation of the mean magnetic field is caused by the $\alpha\Omega$ dynamo. For a rotating nonhelical turbulence the δ term in the mean electromotive force describes the $\Omega \times \mathbf{J}$ effect which causes a generation of the mean magnetic field if rotation is a nonuniform (see [11–15]), where Ω is the angular velocity and \mathbf{J} is the mean electric current.

For a nonrotating and nonhelical turbulence the α effect and the $\Omega \times \mathbf{J}$ effect vanish. However, a mean magnetic field can be generated in a nonrotating and nonhelical turbulence with an imposed mean velocity shear due to the “shear-current” effect [16], described by the δ term in the mean electromotive force. In order to elucidate the physics of the shear-current effect, we compare the α effect in the $\alpha\Omega$ dynamo with the δ term caused by the shear-current effect. The α term in the mean electromotive force, which is responsible for the generation of the mean magnetic field, reads $\mathcal{E}_i^\alpha \equiv \alpha \bar{B}_i \propto -(\Omega \cdot \Lambda) \bar{B}_i$ (see, e.g., [3,15], where $\Lambda = \nabla \langle \mathbf{u}^2 \rangle / \langle \mathbf{u}^2 \rangle$) determines one of the inhomogeneities of the turbulence. The δ term in the electromotive force caused by the shear-current effect is given by $\mathcal{E}_i^\delta \equiv -[\delta \times (\nabla \times \bar{\mathbf{B}})]_i \propto -(\bar{\mathbf{W}} \cdot \nabla) \bar{B}_i$ [see Eq. (8) below and [16]], where the δ term is proportional to the mean vorticity $\bar{\mathbf{W}} = \nabla \times \bar{\mathbf{U}}$. The mean vorticity $\bar{\mathbf{W}}$ in the shear-current dynamo plays the role of a differential rotation and the inhomogeneity of the mean magnetic field plays the role of the inhomogeneity of turbulence. During the generation of the mean magnetic field in both cases (in the $\alpha\Omega$ dynamo and in the shear-current dynamo), the mean electric current along the original mean magnetic field arises. The α effect is related to the hydrodynamic helicity $\propto (\Omega \cdot \Lambda)$ in an inhomogeneous turbulence. The deformations of the mag-

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netic field lines are caused by upward and downward rotating turbulent eddies in the $\alpha\Omega$ dynamo. Since the turbulence is inhomogeneous (which breaks a symmetry between the upward and downward eddies), their total effect on the mean magnetic field does not vanish and it creates the mean electric current along the original mean magnetic field.

In a turbulent flow with an imposed mean velocity shear, the inhomogeneity of the original mean magnetic field breaks a symmetry between the influence of upward and downward turbulent eddies on the mean magnetic field. The deformations of the magnetic field lines in the shear-current dynamo are caused by upward and downward turbulent eddies which result in the mean electric current along the mean magnetic field and produce the magnetic dynamo.

The shear-current effect was studied in [16] in a kinematic approximation. Kinematic dynamo models predict a field that grows without limit, and they give no estimate of the magnitude for the generated magnetic field. In order to find the magnitude of the field, the nonlinear effects which limit the field growth must be taken into account.

The main goal of this study is to develop a nonlinear theory of the shear-current effect. We demonstrated that the nonlinear shear-current effect is a very important nonlinearity in a mean-field dynamo. During the nonlinear growth of the mean magnetic field, the shear-current effect changes its sign, but there is no quenching of this effect contrary to the quenching of the nonlinear α effect, the nonlinear turbulent magnetic diffusion, etc. The nonlinear shear-current effect determines the level of the saturated mean magnetic field. This paper is organized as follows. First, we discuss qualitatively a mechanism for the shear-current effect (Sec. II). In Sec. III we formulate the governing equations, the assumptions, and the procedure of the derivation of the nonlinear mean electromotive force in a turbulent flow with a mean velocity shear. In Sec. IV we analyze the coefficients defining the mean electromotive force for a shear-free turbulence and for a sheared turbulence, and consider the implications of the obtained results of the mean-field magnetic dynamo. The nonlinear saturation of the mean magnetic field and astrophysical applications of the obtained results are discussed in Sec. V.

II. SHEAR-CURRENT EFFECT

In order to describe the shear-current effect we need to determine the mean electromotive force. The general form of the mean electromotive force in a turbulent flow with a mean velocity shear can be obtained even from simple symmetry reasoning. Indeed, the mean electromotive force can be written in the form

$$\mathcal{E}_i = a_{ij}\bar{B}_j + b_{ijk}\bar{B}_{j,k}, \quad (3)$$

where $\bar{B}_{j,i} = \nabla_i \bar{B}_j$ and we neglected terms $\sim O(\nabla^2 \bar{B}_k)$. Following [17] we rewrite Eq. (3) for the mean electromotive force in the form of Eq. (2) with

$$\alpha_{ij}(\bar{\mathbf{B}}) = \frac{1}{2}(a_{ij} + a_{ji}), \quad V_k^{\text{eff}}(\bar{\mathbf{B}}) = \frac{1}{2}\varepsilon_{kji}a_{ij}, \quad (4)$$

$$\eta_{ij}(\bar{\mathbf{B}}) = \frac{1}{4}(\varepsilon_{ikp}b_{jkp} + \varepsilon_{jkp}b_{ikp}), \quad \delta_i = \frac{1}{4}(b_{jji} - b_{jij}), \quad (5)$$

$$\kappa_{ijk}(\bar{\mathbf{B}}) = -\frac{1}{2}(b_{ijk} + b_{ikj}), \quad (6)$$

where we used the identity $\bar{B}_{j,i} = (\partial \hat{B})_{ij} + \varepsilon_{ijm}(\nabla \times \bar{\mathbf{B}})_m/2$. Note that the separation of terms in Eqs. (3)–(6) is not unique, because a gradient term can always be added to the electromotive force. Let us consider a homogeneous, nonrotating, and nonhelical turbulence. Then in the kinematic approximation the tensor a_{ij} vanishes. This implies that $\alpha_{ij} = 0$ and $V_k^{\text{eff}} = 0$. The mean electromotive force \mathcal{E} is a true vector, whereas the mean magnetic field $\bar{\mathbf{B}}$ is a pseudovector. Thus, the tensor b_{ijk} is a pseudotensor [see Eq. (3)]. For homogeneous, isotropic, and nonhelical turbulence the tensor $b_{ijk} = \eta_T \varepsilon_{ijk}$, where $\eta_T = u_0 l_0/3$ is the coefficient of isotropic turbulent magnetic diffusion and u_0 is the characteristic turbulent velocity in the maximum scale of turbulent motions, l_0 . In a turbulent flow with an imposed mean velocity shear, the tensor b_{ijk} depends on the true tensor $\nabla_j \bar{U}_i$. In this case turbulence is anisotropic. The tensor $\nabla_j \bar{U}_i$ can be written as a sum of the symmetric and antisymmetric parts—i.e., $\nabla_j \bar{U}_i = (\partial \hat{U})_{ij} - (1/2)\varepsilon_{ijk} \bar{W}_k$, where $(\partial \hat{U})_{ij} = (\nabla_i \bar{U}_j + \nabla_j \bar{U}_i)/2$ is the true tensor and the mean vorticity $\bar{\mathbf{W}}$ is a pseudovector. Now we take into account the effect which is linear in $\nabla_j \bar{U}_i$. Thus, the pseudotensor b_{ijk} in the kinematic approximation has the form

$$b_{ijk} = \eta_T \varepsilon_{ijk} + l_0^2 [D_1 \varepsilon_{ijm} (\partial \hat{U})_{mk} + D_2 \varepsilon_{ikm} (\partial \hat{U})_{mj} + D_3 \varepsilon_{jkm} (\partial \hat{U})_{mi} + D_4 \delta_{ij} \bar{W}_k + D_5 \delta_{ik} \bar{W}_j], \quad (7)$$

where D_k are the unknown coefficients, δ_{ij} is the Kronecker tensor, and the term $\propto \delta_{jk} \bar{W}_i$ vanishes since $\nabla \cdot \bar{\mathbf{B}} = 0$ [see Eq. (3)]. Using Eqs. (4)–(6) we determine the turbulent coefficients defining the mean electromotive force for a homogeneous and nonhelical turbulence with a mean velocity shear:

$$\eta_{ij} = \eta_T \delta_{ij} - 2l_0^2 \eta_0 (\partial \hat{U})_{ij}, \quad \delta = l_0^2 \delta_0 \bar{\mathbf{W}}, \quad (8)$$

$$\kappa_{ijk} = l_0^2 [\kappa_1 \delta_{ij} \bar{W}_k + \kappa_2 \varepsilon_{ijm} (\partial \hat{U})_{mk}], \quad (9)$$

where $\eta_0 = (D_1 - D_2 - 2D_3)/4$, $\delta_0 = (D_4 - D_5)/2$, $\kappa_1 = -(D_4 + D_5)$, and $\kappa_2 = -(D_1 + D_2)$. The second term in the tensor η_{ij} describes an anisotropic part of turbulent magnetic diffusion caused by the mean velocity shear, while the first term in the tensor η_{ij} is the isotropic contribution to turbulent magnetic diffusion. The δ term for the mean electromotive force describes the shear-current effect which can cause the mean-field magnetic dynamo. Indeed, consider a homogeneous divergence-free turbulence with a mean velocity shear, $\bar{\mathbf{U}} = (0, Sx, 0)$ and $\bar{\mathbf{W}} = (0, 0, S)$. Let us study a simple case when the mean magnetic field is $\bar{\mathbf{B}} = (\bar{B}_x(z), \bar{B}_y(z), 0)$. The mean magnetic field in the kinematic approximation is determined by

$$\frac{\partial \bar{B}_x}{\partial t} = -Sl_0^2 \sigma_0 \bar{B}_y'' + \eta_T \bar{B}_x'', \quad (10)$$

$$\frac{\partial \bar{B}_y}{\partial t} = S\bar{B}_x + \eta_T \bar{B}_y'', \quad (11)$$

where $\bar{B}'' = \partial^2 \bar{B} / \partial z^2$ and

$$\sigma_0 = \delta_0 - \eta_0 - \frac{\kappa_1}{2} - \frac{\kappa_2}{4} = \frac{1}{2}(D_2 + D_3 + 2D_4). \quad (12)$$

In Eq. (11) we took into account that the characteristic spatial scale L_B of the mean magnetic field variations is much larger than the maximum scale of turbulent motions, l_0 . Equation (12) was obtained in [16] in the kinematic approximation. We seek a solution of Eqs. (10) and (11) in the form $\propto \exp(\gamma t + iK_z z)$, where the growth rate γ of the magnetic dynamo instability is given by

$$\gamma = Sl_0 K_z \sqrt{\sigma_0} - \eta_T K_z^2. \quad (13)$$

The first term ($\propto S\bar{B}_x$) on right-hand side (RHS) of Eq. (11) describes the shear motions. This effect is similar to the differential rotation because $\nabla \times (\bar{\mathbf{U}} \times \bar{\mathbf{B}}) = S\bar{B}_x \mathbf{e}_y$. The magnetic dynamo instability is determined by a coupling between the components of the mean magnetic field. In particular, the inhomogeneous magnetic field \bar{B}_y generates the field \bar{B}_x due to the shear-current effect [described by the first term on the RHS of Eq. (10)]. This is similar to the α effect. On the other hand, the field \bar{B}_x generates the field \bar{B}_y due to the pure shear effect [described by the first term on the RHS of Eq. (11)], like the differential rotation in $\alpha\Omega$ dynamo. It follows from Eqs. (10) and (11) that for the shear-current dynamo, $\bar{B}_x / \bar{B}_y \sim l_0 / L_B \ll 1$. Note that in the $\alpha\Omega$ dynamo, the poloidal component of the mean magnetic field is much smaller than the toroidal field.

The magnetic dynamo instability due to the shear-current effect is different from that for the $\alpha\Omega$ dynamo. Indeed, the dynamo mechanism due to the shear-current effect acts even in homogeneous nonhelical turbulence, while the α effect vanishes for homogeneous turbulence.

The shear-current effect was studied in [16] in the kinematic approximation using two different methods: the τ approximation (the Orszag third-order closure procedure) and the stochastic calculus (the path integral representation of the solution of the induction equation, Feynman-Kac formula, and Cameron-Martin-Girsanov theorem). The δ term in the electromotive force which is responsible for the shear-current effect was also independently found in [18] in a problem of a screw dynamo using the modified second-order correlation approximation.

III. GOVERNING EQUATIONS AND THE PROCEDURE OF THE DERIVATION OF THE NONLINEAR EFFECTS

Now let us develop a nonlinear theory of the shear-current effect. In order to derive equations for the nonlinear coeffi-

cients defining the mean electromotive force in a homogeneous turbulence with a mean velocity shear, we will use a mean-field approach in which the magnetic and velocity fields are divided into mean and fluctuating parts, where the fluctuating parts have zero mean values. The procedure of the derivation of equation for the nonlinear mean electromotive force is as follows (for details, see Appendix A). We consider the case of large hydrodynamic and magnetic Reynolds numbers. The momentum equation and the induction equation for the turbulent fields are given by

$$\begin{aligned} \frac{\partial \mathbf{u}(t, \mathbf{x})}{\partial t} = & -\frac{\nabla p_{\text{tot}}}{\rho} + \frac{1}{\mu\rho} [(\mathbf{b} \cdot \nabla) \bar{\mathbf{B}} + (\bar{\mathbf{B}} \cdot \nabla) \mathbf{b}] - (\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} \\ & - (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} + \mathbf{u}^N + \mathbf{F}, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial \mathbf{b}(t, \mathbf{x})}{\partial t} = & (\bar{\mathbf{B}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{B}} - (\bar{\mathbf{U}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{U}} \\ & + \mathbf{b}^N, \end{aligned} \quad (15)$$

where \mathbf{u} and \mathbf{b} are fluctuations of velocity and magnetic field, respectively, $\bar{\mathbf{B}}$ is the mean magnetic field, $\bar{\mathbf{U}}$ is the mean velocity field, ρ is the fluid density, μ is the magnetic permeability of the fluid, \mathbf{F} is a random external stirring force, \mathbf{u}^N and \mathbf{b}^N are the nonlinear terms which include the molecular dissipative terms, $p_{\text{tot}} = p + \mu^{-1}(\bar{\mathbf{B}} \cdot \mathbf{b})$ are the fluctuations of total pressure, and p are the fluctuations of fluid pressure. The velocity \mathbf{u} satisfies the equation $\nabla \cdot \mathbf{u} = 0$. Hereafter we omit μ in the equations; i.e., we include $\mu^{-1/2}$ in the definition of the magnetic field. We study the effect of a mean velocity shear on the mean electromotive force.

Using Eqs. (14) and (15) written in a Fourier space we derive equations for the correlation functions of the velocity field $f_{ij} = \langle u_i u_j \rangle$, the magnetic field $h_{ij} = \langle b_i b_j \rangle$, and the cross-helicity $g_{ij} = \langle b_i u_j \rangle$. The equations for these correlation functions are given by Eqs. (A1)–(A3) in Appendix A. We split the tensors f_{ij} , h_{ij} , and g_{ij} into nonhelical, f_{ij} , and helical, $f_{ij}^{(H)}$, parts. The helical part of the tensor $h_{ij}^{(H)}$ for magnetic fluctuations depends on the magnetic helicity, and it is determined by the dynamic equation which follows from the magnetic helicity conservation arguments (see, e.g., [19–24]). The characteristic time of evolution of the nonhelical part of the magnetic tensor h_{ij} is of the order of the turbulent correlation time $\tau_0 = l_0 / u_0$, while the relaxation time of the helical part of the magnetic tensor $h_{ij}^{(H)}$ is of the order of $\tau_0 \text{Rm}$ (see, e.g., [22]), where $\text{Rm} = l_0 u_0 / \eta \gg 1$ is the magnetic Reynolds number and u_0 is the characteristic turbulent velocity in the maximum scale l_0 of turbulent motions.

Then we split the nonhelical parts of the correlation functions f_{ij} , h_{ij} , and g_{ij} into symmetric and antisymmetric tensors with respect to the wave vector \mathbf{k} , e.g., $f_{ij} = f_{ij}^{(s)} + f_{ij}^{(a)}$, where the tensor $f_{ij}^{(s)} = [f_{ij}(\mathbf{k}) + f_{ij}(-\mathbf{k})] / 2$ describes the symmetric part of the tensor and $f_{ij}^{(a)} = [f_{ij}(\mathbf{k}) - f_{ij}(-\mathbf{k})] / 2$ determines the antisymmetric part of the tensor.

Equations for the second moments contain higher moments, and a problem of closing the equations for the higher moments arises. Various approximate methods have been proposed for the solution of problems of this type (see, e.g.,

[25–27]). The simplest procedure is the τ approximation, which is widely used in the theory of kinetic equations and in passive scalar turbulence and magnetohydrodynamic turbulence (see, e.g., [27–33]). This procedure allows us to express the deviations of the third moments from the background turbulence in terms of the corresponding deviations of the second moments, e.g.,

$$f_{ij}^N - f_{ij}^{N(0)} = -(f_{ij} - f_{ij}^{(0)})/\tau(k), \quad (16)$$

where the term f_{ij}^N is related with the third moment (see Appendix A). The superscript (0) corresponds to the background turbulence (with $\bar{\mathbf{B}}=\mathbf{0}$), and $\tau(k)$ is the characteristic relaxation time of the statistical moments. We applied the τ approximation only for the nonhelical part h_{ij} of the tensor of magnetic fluctuations.

In this study we consider an intermediate nonlinearity which implies that the mean magnetic field is not strong enough in order to affect the correlation time of the turbulent velocity field. The theory for a very strong mean magnetic field can be corrected after taking into account a dependence of the correlation time of the turbulent velocity field on the mean magnetic field. We assume that the characteristic time of variation of the mean magnetic field $\bar{\mathbf{B}}$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales (which corresponds to the mean-field approach). This allows us to get a stationary solution for the equations for the second moments f_{ij} , h_{ij} , and g_{ij} . For the integration in \mathbf{k} space of these second moments we have to specify a model for the background turbulence (with $\bar{\mathbf{B}}=\mathbf{0}$). We use the following model for the background homogeneous and isotropic turbulence: $f_{ij}^{(0)}(\mathbf{k}) = \langle \mathbf{u}^2 \rangle^{(0)} W(k) (\delta_{ij} - k_{ij})$, $h_{ij}^{(0)}(\mathbf{k}) = \langle \mathbf{b}^2 \rangle^{(0)} W(k) \times (\delta_{ij} - k_{ij})$, and $g_{ij}^{(0)}(\mathbf{k}) = 0$, where $k_{ij} = k_i k_j / k^2$, $W(k) = -[d\bar{\tau}(k)/dk]/8\pi k^2$, $\tau(k) = 2\tau_0 \bar{\tau}(k)$, $\bar{\tau}(k) = (k/k_0)^{1-q}$, $1 < q < 3$ is the exponent of the kinetic energy spectrum (e.g., $q=5/3$ for the Kolmogorov spectrum), $k_0 = 1/l_0$, and $\tau_0 = l_0/u_0$, $\int f_{ij}^{(0)}(\mathbf{k}) d\mathbf{k} = \langle \mathbf{u}^2 \rangle^{(0)}/3 \delta_{ij}$, and $\int h_{ij}^{(0)}(\mathbf{k}) d\mathbf{k} = \langle \mathbf{b}^2 \rangle^{(0)}/3 \delta_{ij}$.

Using the derived equations for the second moments f_{ij} , h_{ij} , and g_{ij} we calculate the mean electromotive force $\mathcal{E}_i = \int \tilde{\mathcal{E}}_i(\mathbf{k}) d\mathbf{k}$, where $\tilde{\mathcal{E}}_i(\mathbf{k}) = \varepsilon_{imn} g_{nm}^{(s)}(\mathbf{k})$. For a turbulence with a mean velocity shear the coefficients defining the mean electromotive force are the sum of contributions arising from a shear-free turbulence and sheared turbulence (see Sec. IV).

IV. NONLINEAR MEAN-FIELD DYNAMO IN A TURBULENCE WITH A MEAN VELOCITY SHEAR

First, let us consider a shear-free nonrotating homogeneous and nonhelical turbulence. Using Eqs. (3)–(6), (A35), and (A36) we derive equations for the mean electromotive force. The coefficients defining the mean electromotive force for a shear-free turbulence in a dimensionless form are given by

$$\alpha_{ij}^{(m)} = \alpha^{(m)}(\bar{\mathbf{B}}) \delta_{ij}, \quad (17)$$

$$\mathbf{V}^{\text{eff}} = \mathbf{V}_A(\bar{\mathbf{B}}) + \tilde{\eta}(\bar{\mathbf{B}}) \frac{(\bar{\mathbf{B}} \cdot \nabla) \bar{\mathbf{B}}}{\bar{\mathbf{B}}^2}, \quad (18)$$

$$\eta_{ij} = \eta_A(\bar{\mathbf{B}}) \delta_{ij}, \quad (19)$$

where $\tilde{\eta}(\bar{\mathbf{B}}) = -(1/2)(1+\epsilon)A_2^{(1)}(4\bar{\mathbf{B}})$, the functions $\eta_A(\bar{\mathbf{B}})$ and $\mathbf{V}_A(\bar{\mathbf{B}})$ are determined by Eqs. (22) and (24) respectively, the functions $A_k^{(1)}(y)$ are determined by Eqs. (C1) in Appendix C, and the parameter $\epsilon = \langle \mathbf{b}^2 \rangle^{(0)}/\langle \mathbf{u}^2 \rangle^{(0)}$ is the ratio of the magnetic and kinetic energies in the background turbulence. The function $\alpha^{(m)}(\bar{\mathbf{B}}) = \chi^{(c)}(\bar{\mathbf{B}}) \phi^{(m)}(\sqrt{8\bar{\mathbf{B}}})$ is the magnetic part of the α effect, where $\phi^{(m)}(y) = (3/y^2)(1 - \arctan y/y)$ is the quenching function of the magnetic part of the α effect (see [31,34]), and the dimensionless function $\chi^{(c)}(\bar{\mathbf{B}}) = (\tau/3\mu\rho u_0) \langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle$. The function $\chi^{(c)}(\bar{\mathbf{B}})$ is determined by the dynamic equation which follows from the magnetic helicity conservation arguments (see, e.g., [19–24]). Note that in a homogeneous and nonhelical background turbulence the hydrodynamic part $\alpha_{ij}^{(u)}$ of the α effect vanishes. In a turbulence without a uniform rotation or a mean velocity shear, the $\delta(\bar{\mathbf{B}})$ term and the $\kappa_{ijk}(\bar{\mathbf{B}})$ term in the mean electromotive force vanish.

We adopt here the dimensionless form of the mean dynamo equations; in particular, length is measured in units of L , time is measured in units of L^2/η_T , $\bar{\mathbf{B}}$ is measured in units of the equipartition energy $\bar{B}_{\text{eq}} = \sqrt{\mu\rho}u_0$, and the nonlinear turbulent magnetic diffusion coefficients are measured in units of the characteristic value of the turbulent magnetic diffusivity $\eta_T = l_0 u_0/3$. Note that $L \sim L_B$, where L_B is the characteristic scale of the mean magnetic field variations.

Now we consider a small-scale homogeneous turbulence with a mean velocity shear, $\bar{\mathbf{U}} = Sx\mathbf{e}_y$ and $\bar{\mathbf{W}} = Se_z$. In Cartesian coordinates the mean magnetic field $\bar{\mathbf{B}} = B(x, z)\mathbf{e}_y + \nabla \times [A(x, z)\mathbf{e}_y]$ is determined by the dimensionless dynamo equations

$$\begin{aligned} \frac{\partial A}{\partial t} = & \left(\frac{l_0}{L} \right)^2 S_* \sigma_0(\bar{\mathbf{B}}) (\hat{\mathbf{W}} \cdot \nabla) B + \alpha^{(m)}(\bar{\mathbf{B}}) B - (\mathbf{V}_A(\bar{\mathbf{B}}) \cdot \nabla) A \\ & + \eta_A(\bar{\mathbf{B}}) \Delta A, \end{aligned} \quad (20)$$

$$\frac{\partial B}{\partial t} = -S_* (\hat{\mathbf{W}} \cdot \nabla) A + \nabla \cdot [\eta_B(\bar{\mathbf{B}}) \nabla B], \quad (21)$$

where $S_* = SL^2/\eta_T$ and $\hat{\mathbf{W}} = \bar{\mathbf{W}}/\bar{W}$, and the function $\sigma_0(\bar{\mathbf{B}})$ is determined below. The nonlinear turbulent magnetic diffusion coefficients and the nonlinear drift velocities of the mean magnetic field are given by

$$\eta_A(\bar{\mathbf{B}}) = A_1^{(1)}(4\bar{\mathbf{B}}) + A_2^{(1)}(4\bar{\mathbf{B}}), \quad (22)$$

$$\eta_B(\bar{\mathbf{B}}) = A_1^{(1)}(4\bar{\mathbf{B}}) + 3(1-\epsilon) \left[A_2^{(1)}(4\bar{\mathbf{B}}) - \frac{1}{2\pi} \bar{A}_2(16\bar{\mathbf{B}}^2) \right], \quad (23)$$

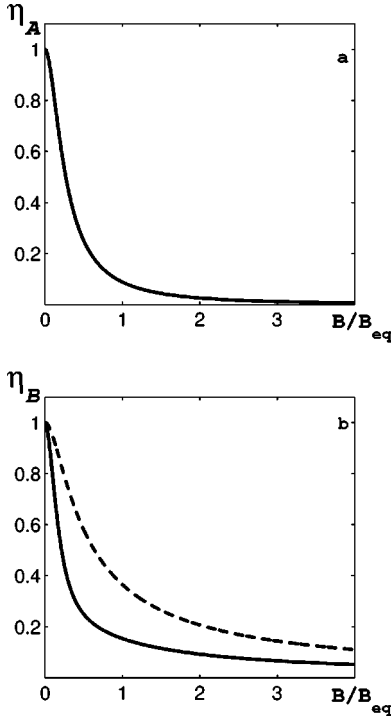


FIG. 1. The nonlinear turbulent magnetic diffusion coefficients η_A (a) and η_B (b) for $\epsilon=0$ (solid lines) and $\epsilon=1$ (dashed lines). The function η_A is independent of the parameter ϵ . The nonlinear turbulent magnetic diffusion coefficients are measured in units of the characteristic value of the turbulent magnetic diffusivity $\eta_T = l_0 u_0 / 3$.

$$V_A(\bar{B}) = -\frac{\Lambda^{(B)}}{2} \left[(2 - 3\epsilon) A_2^{(1)}(4\bar{B}) - (1 - \epsilon) \frac{3}{2\pi} \bar{A}_2(16\bar{B}^2) \right], \quad (24)$$

where $\Lambda^{(B)} = \nabla \bar{B}^2 / \bar{B}^2$, the parameter $0 \leq \epsilon \leq 1$, and the functions $\bar{A}_k(y)$ and $A_k^{(1)}(y)$ are determined by Eqs. (B6) and (C1) in Appendixes B and C. For derivations Eqs. (22)–(24) we used Eqs. (18) and (19). Note that in Eqs. (22)–(24) we neglected small contributions $\sim O[(l_0/L)^2]$ caused by the mean velocity shear. The nonlinear turbulent magnetic diffusion coefficients η_A and η_B and the nonlinear effective drift velocity V_A of mean magnetic field for different value of the parameter ϵ are shown in Figs. 1 and 2. The background magnetic fluctuations caused by the small-scale dynamo result in an increase of the nonlinear turbulent magnetic diffusion coefficient η_B , and they do not affect the nonlinear turbulent magnetic diffusion coefficient η_A (see Fig. 1). On the other hand, the background magnetic fluctuations strongly affect the nonlinear effective drift velocity V_A of the mean magnetic field. In particular, when $\epsilon > 1/2$, the velocity V_A is negative (i.e., it is diamagnetic velocity) which causes a drift of the magnetic field components \bar{B}_x and \bar{B}_z from the regions with a high intensity of the mean magnetic field \bar{B} . When $0 < \epsilon < 1/2$, the effective drift velocity V_A is paramagnetic velocity for a weak mean magnetic field (see Fig. 2). For strong fields $\bar{B} > \bar{B}_{\text{eq}}/2$, the effective drift velocity V_A is dia-

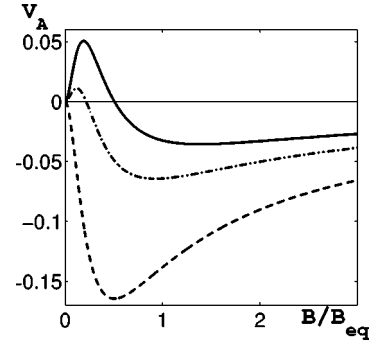


FIG. 2. The nonlinear effective drift velocity V_A of the mean magnetic field for $\epsilon=0$ (solid line), $\epsilon=0.3$ (dash-dotted line), and $\epsilon=1$ (dashed line). The velocity V_A is measured in units of η_T/L .

magnetic for an arbitrary level of background magnetic fluctuations.

The asymptotic formulas for the magnetic part of the α effect, the nonlinear turbulent magnetic diffusion coefficients, and the nonlinear drift velocity of the mean magnetic field for $\bar{B} \ll \bar{B}_{\text{eq}}/4$ are given by

$$\alpha_{ij}^{(m)}(\bar{\mathbf{B}}) = \chi^{(c)}(\bar{\mathbf{B}}) \left(1 - \frac{3\beta^2}{5} \right) \delta_{ij},$$

$$\eta_A(\bar{B}) = 1 - \frac{12}{5}\beta^2, \quad \eta_B(\bar{B}) = 1 - \frac{4}{5}(5 - 4\epsilon)\beta^2,$$

$$V_A(\bar{B}) = \frac{4}{5}(1 - 2\epsilon)\beta^2 \Lambda^{(B)},$$

and for $\bar{B} \gg \bar{B}_{\text{eq}}/4$ they are given by

$$\alpha_{ij}^{(m)}(\bar{\mathbf{B}}) = \chi^{(c)}(\bar{\mathbf{B}}) \frac{3\pi}{2\beta^2} \delta_{ij},$$

$$\eta_A(\bar{B}) = \frac{1}{\beta^2}, \quad \eta_B(\bar{B}) = \frac{2(1 + \epsilon)}{3\beta},$$

$$V_A(\bar{B}) = -\frac{1 + \epsilon}{3\beta} \Lambda^{(B)},$$

where $\beta = \sqrt{8\bar{B}}$.

The nonlinear coefficient $\sigma_0(\bar{B})$ defining the shear-current effect is determined by Eqs. (A51) in Appendix A. The nonlinear dependence of the parameter $\sigma_0(\bar{B})$ is shown in Fig. 3 for different values of the parameter ϵ . The background magnetic fluctuations caused by the small-scale dynamo and described by the parameter ϵ increase the parameter $\sigma_0(\bar{B})$. Note that the parameter $\sigma_0(\bar{B})$ is determined by the contributions from the $\delta(\bar{\mathbf{B}})$ term, the $\eta_{ij}(\bar{\mathbf{B}})$ term, and the $\kappa_{ijk}(\bar{\mathbf{B}})$ term in the mean electromotive force. The asymptotic formula for the parameter $\sigma_0(\bar{B})$ for a weak mean magnetic field $\bar{B} \ll \bar{B}_{\text{eq}}/4$ is given by

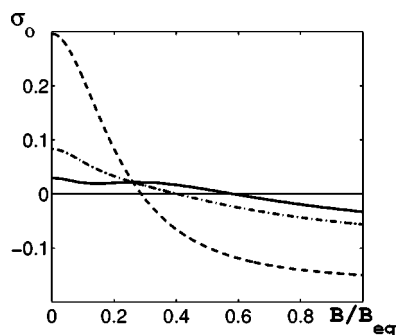


FIG. 3. The dimensionless nonlinear coefficient $\sigma_0(\bar{B})$ defining the shear-current effect for different values of the parameter ϵ : $\epsilon=0$ (solid line), $\epsilon=0.2$ (dash-dotted line), and $\epsilon=1$ (dashed line).

$$\sigma_0(\bar{B}) = \frac{4}{45}(2 - q + 3\epsilon), \quad (25)$$

where q is the exponent of the energy spectrum of the background turbulence. In Eq. (25) we neglected the small contribution $\sim O[(4\bar{B}/\bar{B}_{\text{eq}})^2]$. Equation (25) is in agreement with that obtained in [16] where the case of a weak mean magnetic field and $\epsilon=0$ was considered. Thus, the mean magnetic field is generated due to the shear-current effect, when the exponent of the energy spectrum of the velocity fluctuations is

$$q < 2 + 3\epsilon.$$

Note that the parameter q varies in the range $1 < q < 3$. Therefore, when the level of the background magnetic fluctuations caused by the small-scale dynamo is larger than $1/3$ of the kinetic energy of the velocity fluctuations, the mean magnetic field can be generated due to the shear-current effect for an arbitrary exponent q of the energy spectrum of the velocity fluctuations. For the Kolmogorov turbulence—i.e., when the exponent of the energy spectrum of the background turbulence $q=5/3$ —the parameter $\sigma_0(\bar{B})$ for $\bar{B} \ll \bar{B}_{\text{eq}}/4$ is given by

$$\sigma_0(\bar{B}) = \frac{4}{135}(1 + 9\epsilon), \quad (26)$$

and for $\bar{B} \gg \bar{B}_{\text{eq}}/4$ the parameter $\sigma_0(\bar{B})$ is

$$\sigma_0(\bar{B}) = -\frac{11}{135}(1 + \epsilon). \quad (27)$$

In Eq. (27) we neglected small contribution $\sim O(\bar{B}_{\text{eq}}/4\bar{B})$. It is seen from Eqs. (25)–(27) that the nonlinear coefficient $\sigma_0(\bar{B})$ defining the shear-current effect changes its sign at some value of the mean magnetic field $\bar{B} = \bar{B}_*$. For instance, $\bar{B}_* = 0.6\bar{B}_{\text{eq}}$ for $\epsilon=0$, and $\bar{B}_* = 0.3\bar{B}_{\text{eq}}$ for $\epsilon=1$. The magnitude \bar{B}_* determines the level of the saturated mean magnetic field during its nonlinear evolution (see Sec. V).

Let us determine the threshold for the generation of the mean magnetic field due to the shear-current effect. To this end we introduce the dynamo number in the kinematic approximation:

$$D = \left(\frac{l_0 L S}{\eta_T} \right)^2 \sigma_0(\bar{B}=0). \quad (28)$$

Consider the simple boundary conditions for a layer of thickness $2L$ in the x direction: $B(|x|=1, z)=0$ and $A(|x|=1, z)=0$, where x is measured in units L . Then Eqs. (20) and (21) yield

$$B(t, x, z) = B_0 \exp(\gamma t) \cos(K_x x) \cos(K_z z),$$

$$A(t, x, z) = -B_0 \frac{l_0 \sqrt{\sigma_0}}{L} \exp(\gamma t) \cos(K_x x) \sin(K_z z),$$

with the critical dynamo number $D_{\text{cr}} = \pi^2$, where $\sigma_0(\bar{B}=0) > 0$, the growth rate of the mean magnetic field is $\gamma = \sqrt{DK_z^2 - K_x^2 - K_z^2}$, the wave vector \mathbf{K} is measured in units of L^{-1} , and the growth rate γ is measured in η_T/L^2 . The mean magnetic field is generated when $D > D_{\text{cr}}$. The maximum growth rate of the mean magnetic field, $\gamma_{\text{max}} = D^2/4 - K_x^2$, is attained at $K_z = K_m = \sqrt{D}/2$. The critical dynamo number determines the critical shear of the mean velocity field $S_{\text{cr}} = (\pi/3\sqrt{\sigma_0})(u_0/L)$. The scenario of a nonlinear evolution of the mean magnetic field is discussed in Sec. V.

V. DISCUSSION

In the present paper we developed the nonlinear theory of the shear-current effect in a turbulence with an imposed mean velocity shear. The shear-current effect is associated with the $\bar{\mathbf{W}} \times \bar{\mathbf{J}}$ term in the mean electromotive force and causes the generation of a mean magnetic field even in a nonrotating and nonhelical homogeneous turbulence. The scenario of the mean magnetic field evolution is as follows. In the kinematic stage, the mean magnetic field grows due to the shear-current effect from a very small seeding magnetic field. During the nonlinear growth of the mean magnetic field, the shear-current effect only changes its sign at some value \bar{B}_* of the mean magnetic field. However, there is no quenching of the nonlinear shear-current effect contrary to the quenching of the nonlinear α effect, the nonlinear turbulent magnetic diffusion, etc. The magnitude \bar{B}_* is less than the equipartition field (see below). The background magnetic fluctuations enhance the shear-current effect and result in a decrease of the magnitude \bar{B}_* .

The magnitude \bar{B}_* determines the level of the saturated mean magnetic field. Let us plot the normalized nonlinear dynamo number $D_N^\sigma(\bar{B}) = D^\sigma(\bar{B})/D^\sigma(\bar{B}=0)$ which determines the role of the shear-current effect in the mean magnetic dynamo (see Fig. 4). Here $D^\sigma(\bar{B}) = \sigma_0(\bar{B})/[\eta_A(\bar{B})\eta_B(\bar{B})]$ is the nonlinear dynamo number. At the point $\bar{B} = \bar{B}_*$ the nonlinear effective dynamo number $D_N^\sigma(\bar{B}) = 0$. Depending on the level of the background magnetic fluctuations described by

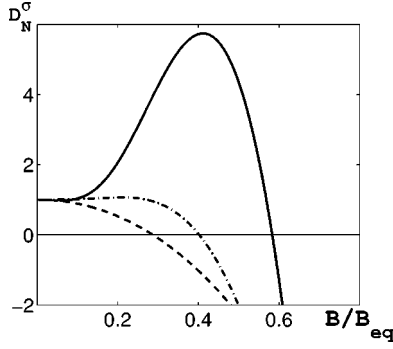


FIG. 4. The normalized nonlinear dynamo number $D_N^\sigma(\bar{B})$ for different values of the parameter ϵ : $\epsilon=0$ (solid line), $\epsilon=0.2$ (dash-dotted line), and $\epsilon=1$ (dashed line).

the parameter ϵ , the saturated mean magnetic field varies from $0.3\bar{B}_{\text{eq}}$ to $0.6\bar{B}_{\text{eq}}$ (see Fig. 4).

Note that the magnetic part of the α effect caused by the magnetic helicity is a purely nonlinear effect. In this study we concentrated on the algebraic nonlinearities (the nonlinear shear-current effect, the nonlinear turbulent magnetic diffusion, the nonlinear effective drift velocity of mean magnetic field) and do not discuss the effect of magnetic helicity (the dynamic nonlinearity, see, e.g., [19–24]) on the nonlinear saturation of the mean magnetic field. This is a subject of an ongoing separate study. Note that the nonlinear shear-current effect can affect the flux of magnetic helicity. However, this remains an open issue.

The shear-current effect may be important in astrophysical objects like accretion disks where mean velocity shear comes together with rotation, so that both the shear-current effect and the α effect might operate. Since the shear-current effect is not quenched contrary to the quenching of the nonlinear α effect, the shear-current effect might be the only surviving effect, and it can explain the dynamics of large-scale magnetic fields in astrophysical bodies with large-scale shearing motions.

APPENDIX A: THE NONLINEAR MEAN ELECTROMOTIVE FORCE IN A TURBULENCE WITH A MEAN VELOCITY SHEAR

We use a mean-field approach whereby the velocity, pressure, and magnetic field are separated into mean and fluctuating parts, where the fluctuating parts have zero mean values. Let us derive the equations for the second moments. In order to exclude the pressure term from the equation of motion (14) we calculate $\nabla \times (\nabla \times \mathbf{u})$. Then we rewrite the obtained equation and Eq. (15) in a Fourier space. We also apply the two-scale approach, e.g., a correlation function

$$\begin{aligned} \langle u_i(\mathbf{x})u_j(\mathbf{y}) \rangle &= \int \langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2) \rangle \exp\{i(\mathbf{k}_1 \cdot \mathbf{x} + \mathbf{k}_2 \cdot \mathbf{y})\} d\mathbf{k}_1 d\mathbf{k}_2 \\ &= \int f_{ij}(\mathbf{k}, \mathbf{K}) \exp(i\mathbf{k} \cdot \mathbf{r} + i\mathbf{K} \cdot \mathbf{R}) d\mathbf{k} d\mathbf{K}, \\ &= \int f_{ij}(\mathbf{k}, \mathbf{R}) \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k}, \end{aligned}$$

where hereafter we omitted argument t in the correlation functions, $f_{ij}(\mathbf{k}, \mathbf{R}) = \hat{L}(u_i; u_j)$,

$$\hat{L}(a; c) = \int \langle a(\mathbf{k} + \mathbf{K}/2)c(-\mathbf{k} + \mathbf{K}/2) \rangle \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K},$$

and $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$, $\mathbf{r} = \mathbf{x} - \mathbf{y}$, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$, and \mathbf{R} and \mathbf{K} correspond to the large scales and \mathbf{r} and \mathbf{k} to the small ones (see, e.g., [35,36]). This implies that we assumed that there exists a separation of scales; i.e., the maximum scale of turbulent motions l_0 is much smaller than the characteristic scale L of inhomogeneities of the mean fields. In particular, this implies that $r \leq l_0 \ll R$. Our final results showed that this assumption is indeed valid. We derive equations for the following correlation functions:

$$f_{ij}(\mathbf{k}, \mathbf{R}) = \hat{L}(u_i; u_j), \quad h_{ij}(\mathbf{k}, \mathbf{R}) = \hat{L}(b_i; b_j),$$

$$g_{ij}(\mathbf{k}, \mathbf{R}) = \hat{L}(b_i; u_j).$$

The equations for these correlation functions are given by

$$\frac{\partial f_{ij}(\mathbf{k})}{\partial t} = i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M)} + I_{ij}^f + I_{ijmn}^\sigma(\bar{\mathbf{U}}) f_{mn} + F_{ij} + F_{ij}^N, \quad (\text{A1})$$

$$\frac{\partial h_{ij}(\mathbf{k})}{\partial t} = -i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M)} + I_{ij}^h + E_{ijmn}^\sigma(\bar{\mathbf{U}}) h_{mn} + h_{ij}^N, \quad (\text{A2})$$

$$\begin{aligned} \frac{\partial g_{ij}(\mathbf{k})}{\partial t} &= i(\mathbf{k} \cdot \bar{\mathbf{B}}) [f_{ij}(\mathbf{k}) - h_{ij}(\mathbf{k}) - h_{ij}^{(H)}] + I_{ij}^g + J_{ijmn}^\sigma(\bar{\mathbf{U}}) g_{mn} \\ &\quad + g_{ij}^N, \end{aligned} \quad (\text{A3})$$

where hereafter we omitted the argument t and \mathbf{R} in the correlation functions and neglected the terms $\sim O(\nabla^2)$.

Here $\Phi_{ij}^{(M)}(\mathbf{k}) = g_{ij}(\mathbf{k}) - g_{ji}(-\mathbf{k})$, $F_{ij}(\mathbf{k}) = \langle \tilde{F}_i(\mathbf{k})u_j(-\mathbf{k}) \rangle + \langle u_i(\mathbf{k})\tilde{F}_j(-\mathbf{k}) \rangle$, and $\tilde{\mathbf{F}}(\mathbf{k}) = \mathbf{k} \times (\mathbf{k} \times \mathbf{F}(\mathbf{k})) / k^2 \rho$. The tensors $I_{ijmn}^\sigma(\bar{\mathbf{U}})$, $E_{ijmn}^\sigma(\bar{\mathbf{U}})$, and $J_{ijmn}^\sigma(\bar{\mathbf{U}})$ are given by

$$\begin{aligned} I_{ijmn}^\sigma(\bar{\mathbf{U}}) &= \left[2k_{iq} \delta_{mp} \delta_{jn} + 2k_{jq} \delta_{im} \delta_{pn} - \delta_{im} \delta_{jq} \delta_{np} - \delta_{iq} \delta_{jn} \delta_{mp} \right. \\ &\quad \left. + \delta_{im} \delta_{jn} k_q \frac{\partial}{\partial k_p} \right] \nabla_p \bar{U}_q, \end{aligned}$$

$$E_{ijmn}^\sigma(\bar{\mathbf{U}}) = (\delta_{im} \delta_{jq} + \delta_{jm} \delta_{iq}) \nabla_n \bar{U}_q,$$

$$\begin{aligned} J_{ijmn}^\sigma(\bar{\mathbf{U}}) &= \left[2k_{jq} \delta_{im} \delta_{pn} - \delta_{im} \delta_{pn} \delta_{jq} + \delta_{jn} \delta_{pm} \delta_{iq} \right. \\ &\quad \left. + \delta_{im} \delta_{jn} k_q \frac{\partial}{\partial k_p} \right] \nabla_p \bar{U}_q, \end{aligned}$$

where δ_{ij} is the Kronecker tensor and $k_{ij} = k_i k_j / k^2$. Equations (A1)–(A3) are written in a frame moving with a local velocity $\bar{\mathbf{U}}$ of the mean flows. In Eqs. (A1) and (A3) we neglected

small terms which are of the order of $O(|\nabla^2 \bar{\mathbf{U}}|)$. The source terms I_{ij}^f, I_{ij}^h and I_{ij}^g which contain the large-scale spatial derivatives of the mean magnetic field are given by

$$I_{ij}^f = \frac{1}{2}(\bar{\mathbf{B}} \cdot \nabla) \Phi_{ij}^{(P)} + [g_{qj}(\mathbf{k})(2P_{in}(k) - \delta_{in}) + g_{qi}(-\mathbf{k})(2P_{jn}(k) - \delta_{jn})] \bar{B}_{n,q} - \bar{B}_{n,q} k_n \Phi_{ijq}^{(P)}, \quad (\text{A4})$$

$$I_{ij}^h = \frac{1}{2}(\bar{\mathbf{B}} \cdot \nabla) \Phi_{ij}^{(P)} - [g_{iq}(\mathbf{k}) \delta_{jn} + g_{jq}(-\mathbf{k}) \delta_{in}] \bar{B}_{n,q} - \bar{B}_{n,q} k_n \Phi_{ijq}^{(P)}, \quad (\text{A5})$$

$$I_{ij}^g = \frac{1}{2}(\bar{\mathbf{B}} \cdot \nabla)(f_{ij} + h_{ij}) + h_{iq}(2P_{jn}(k) - \delta_{jn}) \bar{B}_{n,q} - f_{nj} \bar{B}_{i,n} - \bar{B}_{n,q} k_n (f_{ijq} + h_{ijq}), \quad (\text{A6})$$

where $\nabla = \partial / \partial \mathbf{R}$, $\Phi_{ij}^{(P)}(\mathbf{k}) = g_{ij}(\mathbf{k}) + g_{ji}(-\mathbf{k})$, and $\bar{B}_{i,j} = \nabla_j \bar{B}_i$, the terms f_{ij}^N, h_{ij}^N and g_{ij}^N are determined by the third moments appearing due to the nonlinear terms, $f_{ijq} = (1/2) \partial f_{ij} / \partial k_q$, and similarly for h_{ijq} and $\Phi_{ijq}^{(P)}$. A stirring force in the Navier-Stokes turbulence is an external parameter that determines the background turbulence.

For the derivation of Eqs. (A1)–(A3) we performed several calculations that are similar to the following, which arose, e.g., in computing $\partial g_{ij} / \partial t$. The other calculations follow similar lines and are not given here. Let us define $Y_{ij}(\mathbf{k}, \mathbf{R})$ by

$$Y_{ij}(\mathbf{k}, \mathbf{R}) = i \int (k_p + K_p/2) \bar{B}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R}) \times \langle u_i(\mathbf{k} + \mathbf{K}/2 - \mathbf{Q}) u_j(-\mathbf{k} + \mathbf{K}/2) \rangle d\mathbf{K} d\mathbf{Q}. \quad (\text{A7})$$

Next, we introduce the new variables

$$\begin{aligned} \tilde{\mathbf{k}}_1 &= \mathbf{k} + \mathbf{K}/2 - \mathbf{Q}, & \tilde{\mathbf{k}}_2 &= -\mathbf{k} + \mathbf{K}/2, \\ \tilde{\mathbf{k}} &= (\tilde{\mathbf{k}}_1 - \tilde{\mathbf{k}}_2)/2 = \mathbf{k} - \mathbf{Q}/2, & \tilde{\mathbf{K}} &= \tilde{\mathbf{k}}_1 + \tilde{\mathbf{k}}_2 = \mathbf{K} - \mathbf{Q}. \end{aligned} \quad (\text{A8})$$

Therefore,

$$Y_{ij}(\mathbf{k}, \mathbf{R}) = i \int f_{ij}(\mathbf{k} - \mathbf{Q}/2, \mathbf{K} - \mathbf{Q}) (k_p + K_p/2) \bar{B}_p(\mathbf{Q}) \times \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q}. \quad (\text{A9})$$

Since $|\mathbf{Q}| \ll |\mathbf{k}|$, we use the Taylor expansion

$$f_{ij}(\mathbf{k} - \mathbf{Q}/2, \mathbf{K} - \mathbf{Q}) \approx f_{ij}(\mathbf{k}, \mathbf{K} - \mathbf{Q}) - \frac{1}{2} \frac{\partial f_{ij}(\mathbf{k}, \mathbf{K} - \mathbf{Q})}{\partial k_s} Q_s + O(Q^2), \quad (\text{A10})$$

and the identities

$$[f_{ij}(\mathbf{k}, \mathbf{R}) \bar{B}_p(\mathbf{R})]_{\mathbf{K}} = \int f_{ij}(\mathbf{k}, \mathbf{K} - \mathbf{Q}) \bar{B}_p(\mathbf{Q}) d\mathbf{Q},$$

$$\begin{aligned} \nabla_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{B}_p(\mathbf{R})] &= \int i K_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{B}_p(\mathbf{R})]_{\mathbf{K}} \\ &\times \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}. \end{aligned} \quad (\text{A11})$$

Therefore, Eqs. (A9)–(A11) yield

$$\begin{aligned} Y_{ij}(\mathbf{k}, \mathbf{R}) &\approx [i(\mathbf{k} \cdot \bar{\mathbf{B}}) + (1/2)(\bar{\mathbf{B}} \cdot \nabla)] f_{ij}(\mathbf{k}, \mathbf{R}) \\ &- \frac{1}{2} k_p \frac{\partial f_{ij}(\mathbf{k})}{\partial k_s} \bar{B}_{p,s}. \end{aligned} \quad (\text{A12})$$

We took into account that in Eq. (A3) the terms with symmetric tensors with respect to the indexes i and j do not contribute to the mean electromotive force because $\mathcal{E}_m = \varepsilon_{mji} g_{ij}$. In Eqs. (A9)–(A11) we neglected the second and higher derivatives over \mathbf{R} . For the derivation of Eqs. (A1)–(A3) we also used the identity

$$\begin{aligned} ik_i \int f_{ij} \left(\mathbf{k} - \frac{1}{2} \mathbf{Q}, \mathbf{K} - \mathbf{Q} \right) \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} \\ = -\frac{1}{2} \bar{U}_p \nabla_i f_{ij} + \frac{1}{2} f_{ij} \nabla_i \bar{U}_p - \frac{i}{4} (\nabla_s \bar{U}_p) \left(\nabla_i \frac{\partial f_{ij}}{\partial k_s} \right) \\ + \frac{i}{4} \left(\frac{\partial f_{ij}}{\partial k_s} \right) (\nabla_s \nabla_i \bar{U}_p). \end{aligned} \quad (\text{A13})$$

To derive Eq. (A13) we multiply the equation $\nabla \cdot \mathbf{u} = 0$ [written in \mathbf{k} space for $u_i(\mathbf{k}_1 - \mathbf{Q})$] by $u_j(\mathbf{k}_2) \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R})$, integrate over \mathbf{K} and \mathbf{Q} , and average over ensemble of velocity fluctuations. Here $\mathbf{k}_1 = \mathbf{k} + \mathbf{K}/2$ and $\mathbf{k}_2 = -\mathbf{k} + \mathbf{K}/2$. This yields

$$\begin{aligned} \int i \left(k_i + \frac{1}{2} K_i - Q_i \right) \left\langle u_i \left(\mathbf{k} + \frac{1}{2} \mathbf{K} - \mathbf{Q} \right) u_j \left(-\mathbf{k} + \frac{1}{2} \mathbf{K} \right) \right\rangle \\ \times \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} = 0. \end{aligned} \quad (\text{A14})$$

Now we introduce the new variables $\tilde{\mathbf{k}}_1$ and $\tilde{\mathbf{k}}_2$ determined by Eq. (A8). This allows us to rewrite Eq. (A14) in the form

$$\begin{aligned} \int i \left(k_i + \frac{1}{2} K_i - Q_i \right) f_{ij} \left(\mathbf{k} - \frac{1}{2} \mathbf{Q}, \mathbf{K} - \mathbf{Q} \right) \bar{U}_p(\mathbf{Q}) \\ \times \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} = 0. \end{aligned} \quad (\text{A15})$$

Since $|\mathbf{Q}| \ll |\mathbf{k}|$, we use the Taylor expansion (A10), and we also use the following identities, which are similar to Eq. (A11):

$$[f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})]_{\mathbf{K}} = \int f_{ij}(\mathbf{k}, \mathbf{K} - \mathbf{Q}) \bar{U}_p(\mathbf{Q}) d\mathbf{Q},$$

$$\begin{aligned} \nabla_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})] &= \int i K_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})]_{\mathbf{K}} \\ &\times \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}. \end{aligned} \quad (\text{A16})$$

Therefore, Eq. (A15) yields Eq. (A13).

Now we split all tensors into nonhelical, f_{ij} , and helical, $f_{ij}^{(H)}$, parts. Note that the helical part of the tensor of magnetic fluctuations, $h_{ij}^{(H)}$, depends on the magnetic helicity and is not determined by Eq. (A2). The equation for the helical part of

the tensor of magnetic fluctuations, $h_{ij}^{(H)}$, follows from the magnetic helicity conservation arguments (see, e.g., [19–24]).

In this study we use the τ approximation [see Eq. (16)]. The τ approximation is in general similar to eddy-damped quasnormal Markovian (EDQNM) approximation. However, some principal difference exists between these two approaches (see [26,27]). The EDQNM closures do not relax to equilibrium, and this procedure does not describe properly the motions in the equilibrium state in contrast to the τ approximation. Within the EDQNM theory, there is no dynamically determined relaxation time and no slightly perturbed steady state can be approached [27]. In the τ approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from equilibrium [27]. As follows from the analysis by [27] the τ approximation describes the relaxation to the equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

1. Shear-free homogeneous turbulence

Consider a turbulence without a mean velocity shear; i.e., we omit the tensors $I_{ijmn}^\sigma(\bar{\mathbf{U}})$, $E_{ijmn}^\sigma(\bar{\mathbf{U}})$, and $J_{ijmn}^\sigma(\bar{\mathbf{U}})$ in Eqs. (A1)–(A3). First we solve Eqs. (A1)–(A3) neglecting the sources I_{ij}^f , I_{ij}^h , and I_{ij}^g with the large-scale spatial derivatives. Then we will take into account the terms with the large-scale spatial derivatives by perturbations. We start with Eqs. (A1)–(A3) written for nonhelical parts of the tensors and then consider Eqs. (A1)–(A3) for helical parts of the tensors.

Thus, we subtract Eqs. (A1)–(A3) written for background turbulence (for $\bar{\mathbf{B}}=0$) from those for $\bar{\mathbf{B}} \neq 0$. Then we use the τ approximation and neglect the terms with the large-scale spatial derivatives. Next, we assume that $\eta k^2 \ll \tau^{-1}$ and $\nu k^2 \ll \tau^{-1}$ for the inertial range of turbulent flow, and we also assume that the characteristic time of variation of the mean magnetic field $\bar{\mathbf{B}}$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. Thus, we arrive to the following steady-state solution of the obtained equations:

$$\hat{f}_{ij}(\mathbf{k}) \approx f_{ij}^{(0)}(\mathbf{k}) + i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})\hat{\Phi}_{ij}^{(M)}(\mathbf{k}), \quad (\text{A17})$$

$$\hat{h}_{ij}(\mathbf{k}) \approx h_{ij}^{(0)}(\mathbf{k}) - i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})\hat{\Phi}_{ij}^{(M)}(\mathbf{k}), \quad (\text{A18})$$

$$\hat{g}_{ij}(\mathbf{k}) \approx i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})[\hat{f}_{ij}(\mathbf{k}) - \hat{h}_{ij}(\mathbf{k})], \quad (\text{A19})$$

where \hat{f}_{ij} , \hat{h}_{ij} , and \hat{g}_{ij} are solutions without the sources I_{ij}^f , I_{ij}^h , and I_{ij}^g .

Now we split all correlation functions into symmetric and antisymmetric parts with respect to the wave number \mathbf{k} , e.g., $f_{ij} = f_{ij}^{(s)} + f_{ij}^{(a)}$, where $f_{ij}^{(s)} = [f_{ij}(\mathbf{k}) + f_{ij}(-\mathbf{k})]/2$ is the symmetric part and $f_{ij}^{(a)} = [f_{ij}(\mathbf{k}) - f_{ij}(-\mathbf{k})]/2$ is the antisymmetric part, and similarly for other tensors. Thus, Eqs. (A17)–(A19) yield

$$\hat{f}_{ij}^{(s)}(\mathbf{k}) \approx \frac{1}{1+2\psi}[(1+\psi)f_{ij}^{(0)}(\mathbf{k}) + \psi h_{ij}^{(0)}(\mathbf{k})], \quad (\text{A20})$$

$$\hat{h}_{ij}^{(s)}(\mathbf{k}) \approx \frac{1}{1+2\psi}[\psi f_{ij}^{(0)}(\mathbf{k}) + (1+\psi)h_{ij}^{(0)}(\mathbf{k})], \quad (\text{A21})$$

$$\hat{g}_{ij}^{(a)}(\mathbf{k}) \approx \frac{i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})}{1+2\psi}[f_{ij}^{(0)}(\mathbf{k}) - h_{ij}^{(0)}(\mathbf{k})], \quad (\text{A22})$$

where $\psi(\mathbf{k}) = 2(\tau\mathbf{k} \cdot \bar{\mathbf{B}})^2$. The correlation functions $\hat{f}_{ij}^{(a)}(\mathbf{k})$, $\hat{h}_{ij}^{(a)}(\mathbf{k})$, and $\hat{g}_{ij}^{(s)}(\mathbf{k})$ vanish if we neglect the large-scale spatial derivatives; i.e., they are proportional to the first-order spatial derivatives. Equations (A20) and (A21) yield

$$\hat{f}_{ij}^{(s)}(\mathbf{k}) + \hat{h}_{ij}^{(s)}(\mathbf{k}) \approx f_{ij}^{(0)}(\mathbf{k}) + h_{ij}^{(0)}(\mathbf{k}), \quad (\text{A23})$$

which is in agreement with the fact that a uniform mean magnetic field performs no work on the turbulence. A uniform mean magnetic field can only redistribute the energy between hydrodynamic fluctuations and magnetic fluctuation. A change of the total energy of fluctuations is caused by a nonuniform mean magnetic field.

Next, we take into account the large-scale spatial derivatives in Eqs. (A1)–(A3) by perturbations. Their effect determines the following steady-state equations for the second moments \tilde{f}_{ij} , \tilde{h}_{ij} , and \tilde{g}_{ij} :

$$\tilde{f}_{ij}^{(a)}(\mathbf{k}) = i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})\tilde{\Phi}_{ij}^{(M,s)}(\mathbf{k}) + \tau I_{ij}^f, \quad (\text{A24})$$

$$\tilde{h}_{ij}^{(a)}(\mathbf{k}) = -i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})\tilde{\Phi}_{ij}^{(M,s)}(\mathbf{k}) + \tau I_{ij}^h, \quad (\text{A25})$$

$$\tilde{g}_{ij}^{(s)}(\mathbf{k}) = i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})(\tilde{f}_{ij}^{(a)}(\mathbf{k}) - \tilde{h}_{ij}^{(a)}(\mathbf{k})) + \tau I_{ij}^g, \quad (\text{A26})$$

where $\tilde{\Phi}_{ij}^{(M,s)} = [\tilde{\Phi}_{ij}^{(M)}(\mathbf{k}) + \tilde{\Phi}_{ij}^{(M)}(-\mathbf{k})]/2$. The solution of Eqs. (A24)–(A26) yields

$$\tilde{\Phi}_{ij}^{(M,s)}(\mathbf{k}) = \frac{\tau}{1+2\psi}\{I_{ij}^g - I_{ji}^g + i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})(I_{ij}^f - I_{ji}^f + I_{ij}^h - I_{ji}^h)\}. \quad (\text{A27})$$

Substituting Eq. (A27) into Eqs. (A24)–(A26) we obtain the final expressions in \mathbf{k} space for the nonhelical parts of the tensors $\tilde{f}_{ij}^{(a)}(\mathbf{k})$, $\tilde{h}_{ij}^{(a)}(\mathbf{k})$, $\tilde{g}_{ij}^{(s)}(\mathbf{k})$, and $\tilde{\Phi}_{ij}^{(M,s)}(\mathbf{k})$. In particular,

$$\begin{aligned} \tilde{\Phi}_{mn}^{(M,s)}(\mathbf{k}) = & \frac{\tau(k)W(k)\langle \mathbf{u}^2 \rangle^{(0)}}{(1+2\psi)^2} [(1+\epsilon)(1+2\psi)(\delta_{nj}\delta_{mk} - \delta_{mj}\delta_{nk} \\ & + k_{nk}\delta_{mj} - k_{mk}\delta_{nj}) - 2(\epsilon+2\psi)(k_{nj}\delta_{mk} \\ & - k_{mj}\delta_{nk})]\bar{\mathbf{B}}_{j,k}. \end{aligned} \quad (\text{A28})$$

The correlation functions $\tilde{f}_{ij}^{(s)}(\mathbf{k})$, $\tilde{h}_{ij}^{(s)}(\mathbf{k})$, and $\tilde{g}_{ij}^{(a)}(\mathbf{k})$ are of the order of $\sim O(\nabla^2)$; i.e., they are proportional to the second-order spatial derivatives. Thus $\hat{f}_{ij} + \tilde{f}_{ij}$ is the nonhelical part of the correlation functions for a shear-free turbulence and similarly for other second moments.

Now we solve Eqs. (A1)–(A3) for the helical parts of the tensors for a shear-free turbulence using the same approach which we used in this section. The steady-state solution of Eqs. (A1) and (A3) for the helical parts of the tensors reads

$$f_{ij}^{(H)}(\mathbf{k}) \approx i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})\Phi_{ij}^{(M,H)}(\mathbf{k}), \quad (\text{A29})$$

$$g_{ij}^{(H)}(\mathbf{k}) \approx i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})[f_{ij}^{(H)}(\mathbf{k}) - h_{ij}^{(H)}(\mathbf{k})], \quad (\text{A30})$$

where $\Phi_{ij}^{(M,H)}(\mathbf{k}) = g_{ij}^{(H)}(\mathbf{k}) - g_{ji}^{(H)}(-\mathbf{k})$. The tensor $h_{ij}^{(H)}$ is determined by the dynamic equation (see, e.g., [19–24]). The solution of Eqs. (A29) and (A30) yields

$$\Phi_{ij}^{(M,H)}(\mathbf{k}) = -\frac{2i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})}{1 + \psi} h_{ij}^{(H)}. \quad (\text{A31})$$

Since $h_{ij}^{(H)}$ is of the order of $O(\nabla)$, we do not need to take into account the source terms with the large-scale spatial derivatives [22].

Now we calculate the mean electromotive force $\mathcal{E}_i(\mathbf{r}=0) = (1/2)\varepsilon_{imn} \int [\Phi_{mn}^{(M,H)}(\mathbf{k}) + \tilde{\Phi}_{mn}^{(M,s)}(\mathbf{k})] d\mathbf{k}$. Thus,

$$\mathcal{E}_i = \varepsilon_{imn} \int \left[\frac{\tau}{1 + 2\psi} [I_{mn}^g + i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})(I_{mn}^f - I_{mn}^h)] - \frac{i\tau(\mathbf{k} \cdot \bar{\mathbf{B}})}{1 + \psi} h_{mn}^{(H)} \right] d\mathbf{k}. \quad (\text{A32})$$

For the integration in \mathbf{k} space of the mean electromotive force we have to specify a model for the background turbulence (with $\bar{\mathbf{B}}=0$); see Sec. III. After integration in \mathbf{k} space we obtain $\mathcal{E}_i = a_{ij}\bar{B}_j + b_{ijk}\bar{B}_{j,k}$, where

$$a_{ij} = -i\varepsilon_{imn} \int \frac{\tau k_j h_{mn}^{(H)}}{1 + \psi} d\mathbf{k} = \chi^{(c)}(\bar{\mathbf{B}}) \left\{ \phi^{(m)}(\beta) \beta_{ij} + \frac{1}{2} [3 - (1 + \beta^2) \phi^{(m)}(\beta)] P_{ij}(\beta) \right\}, \quad (\text{A33})$$

$$b_{ijk} = \frac{1}{2} \eta_T ((1 + \epsilon) \varepsilon_{ijm} [\delta_{km} K_{pp}^{(1)}(\sqrt{2}\beta) - K_{km}^{(1)}(\sqrt{2}\beta)] + 2\varepsilon_{ink} [(1 - \epsilon) \bar{\Psi}_1 \{K_{jn}\} - K_{jn}^{(1)}(\sqrt{2}\beta)]), \quad (\text{A34})$$

$$\bar{\Psi}_1 \{X\} = 3X^{(1)}(\sqrt{2}\beta) - \frac{3}{2\pi} \bar{X}(2\beta^2),$$

$$K_{ij}^{(1)}(\beta) = \frac{3}{2\pi} \int_0^1 \bar{K}_{ij}(a(\bar{\tau})) \bar{\tau} d\bar{\tau} = \frac{3\beta^4}{\pi} \int_\beta^\infty \frac{\bar{K}_{ij}(X^2)}{X^5} dX,$$

and all calculations are made for $q=5/3$, $X^2 = \beta^2(k/k_0)^{2/3} = a = [\beta u_0 k \tau(k)/2]^2$, the function \bar{K}_{ij} is determined by Eq. (B1) in Appendix B, $\epsilon = \langle \mathbf{b}^2 \rangle^{(0)} / \langle \mathbf{u}^2 \rangle^{(0)}$, $\beta = 4\bar{B}/(u_0 \sqrt{2\mu\rho})$, $P_{ij}(\beta) = \delta_{ij} - \beta_{ij}$, $\beta_{ij} = \bar{B}_i \bar{B}_j / \bar{B}^2$, $\phi^{(m)}(\beta) = (3/\beta^2)[1 - \arctan(\beta)/\beta]$, and $\chi^{(c)}(\bar{\mathbf{B}}) \equiv (\tau/3\mu\rho) \langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle$ is related with current helicity. Since a part of the mean electromotive force is determined by the function $a_{ij}(\bar{\mathbf{B}})\bar{B}_j$ and $P_{ij}(\beta)\bar{B}_j = 0$, we can drop the term $\propto P_{ij}(\beta)$ in Eq. (A33). Thus, the equations for a_{ij} and b_{ijk} are given by

$$a_{ij} = \alpha^{(m)}(\bar{\mathbf{B}}) \delta_{ij}, \quad (\text{A35})$$

$$b_{ijk} = \eta_T [b_1 \varepsilon_{ijk} + b_2 \varepsilon_{ijn} \beta_{nk} + b_3 \varepsilon_{ink} \beta_{nj}], \quad (\text{A36})$$

where $\alpha^{(m)}(\bar{\mathbf{B}}) = \chi^{(c)}(\bar{\mathbf{B}}) \phi^{(m)}(\beta)$ and

$$b_1 = A_1^{(1)}(\sqrt{2}\beta) + A_2^{(1)}(\sqrt{2}\beta),$$

$$b_2 = -\frac{1}{2}(1 + \epsilon) A_2^{(1)}(\sqrt{2}\beta),$$

$$b_3 = (1 - \epsilon) \bar{\Psi}_1 \{A_2\} - A_2^{(1)}(\sqrt{2}\beta),$$

the functions $\bar{A}_k(y)$ and $A_k^{(1)}(y)$ being determined by Eqs. (B6) and (C1) in Appendixes B and C. Equations (A35) and (A36) yield Eqs. (17)–(19).

2. Turbulence with a mean velocity shear

Now we study the effect of a mean velocity shear on the mean electromotive force. We take into account the tensors $I_{ijmn}^\sigma(\bar{\mathbf{U}})$, $E_{ijmn}^\sigma(\bar{\mathbf{U}})$ and $J_{ijmn}^\sigma(\bar{\mathbf{U}})$ in Eqs. (A1)–(A3), and we neglect terms $\sim O(\nabla^2)$. The steady-state solution of Eqs. (A1)–(A3) for the nonhelical parts of the tensors for a sheared turbulence reads

$$N_{ijmn}^f(\bar{\mathbf{U}}) f_{mn} = \tau \{ i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M)} + I_{ij}^f \}, \quad (\text{A37})$$

$$N_{ijmn}^h(\bar{\mathbf{U}}) h_{mn} = \tau \{ -i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M)} + I_{ij}^h \}, \quad (\text{A38})$$

$$N_{ijmn}^g(\bar{\mathbf{U}}) g_{mn} = \tau \{ i(\mathbf{k} \cdot \bar{\mathbf{B}}) [f_{ij}(\mathbf{k}) - h_{ij}(\mathbf{k})] + I_{ij}^g \}, \quad (\text{A39})$$

where

$$N_{ijmn}^f(\bar{\mathbf{U}}) = \delta_{im} \delta_{jn} - \tau I_{ijmn}^\sigma,$$

$$N_{ijmn}^h(\bar{\mathbf{U}}) = \delta_{im} \delta_{jn} - \tau E_{ijmn}^\sigma,$$

$$N_{ijmn}^g(\bar{\mathbf{U}}) = \delta_{im} \delta_{jn} - \tau J_{ijmn}^\sigma,$$

and we use the following notation: the total correlation function is $f_{ij} = \bar{f}_{ij} + f_{ij}^\sigma$. Here $\bar{f}_{ij} = \hat{f}_{ij} + \tilde{f}_{ij}$ is the correlation functions for a shear-free turbulence, and the correlation functions f_{ij}^σ determines the effect of a mean velocity shear. The similar notations are for other correlation functions. Now we solve Eqs. (A37)–(A39) by iterations. This yields

$$f_{ij}^\sigma(\mathbf{k}) = \tau \{ I_{ijmn}^\sigma \bar{f}_{mn} + i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M,\sigma)} + I_{ij}^{(f,\sigma)} \}, \quad (\text{A40})$$

$$h_{ij}^\sigma(\mathbf{k}) = \tau \{ E_{ijmn}^\sigma \bar{h}_{mn} - i(\mathbf{k} \cdot \bar{\mathbf{B}}) \Phi_{ij}^{(M,\sigma)} + I_{ij}^{(h,\sigma)} \}, \quad (\text{A41})$$

$$g_{ij}^\sigma(\mathbf{k}) = \tau \{ J_{ijmn}^\sigma \bar{g}_{mn} + i(\mathbf{k} \cdot \bar{\mathbf{B}}) [f_{ij}^\sigma - h_{ij}^\sigma] + I_{ij}^{(g,\sigma)} \}, \quad (\text{A42})$$

where $\Phi_{ij}^{(M,\sigma)}(\mathbf{k}) = g_{ij}^\sigma(\mathbf{k}) - g_{ji}^\sigma(-\mathbf{k})$ and the source terms $I_{ij}^{(f,\sigma)} \equiv I_{ij}^f(g_{ij}^\sigma)$, $I_{ij}^{(h,\sigma)} \equiv I_{ij}^h(g_{ij}^\sigma)$ and $I_{ij}^{(g,\sigma)} \equiv I_{ij}^g(f_{ij}^\sigma, h_{ij}^\sigma)$ are determined by Eqs. (A4)–(A6), where \bar{f}_{ij} , h_{ij} , and g_{ij} are replaced by f_{ij}^σ , h_{ij}^σ , and g_{ij}^σ , respectively. The solution of Eqs. (A40)–(A42) yield equation for the symmetric part $\Phi_{ij}^{(M,\sigma,s)}$ of the tensor:

$$\Phi_{ij}^{(M,\sigma,s)}(\mathbf{k}) = \frac{\tau}{1+2\psi} \{ (J_{ijmn}^\sigma - J_{jimn}^\sigma) \tilde{g}_{mn} + I_{ij}^{(g,\sigma)} - I_{ji}^{(g,\sigma)} + i\tau(\mathbf{k} \cdot \bar{\mathbf{B}}) [(I_{ijmn}^\sigma - I_{jimn}^\sigma) \tilde{f}_{mn} + I_{ij}^{(f,\sigma)} - I_{ji}^{(f,\sigma)} + I_{ji}^{(h,\sigma)} - I_{ij}^{(h,\sigma)}] \}, \quad (A43)$$

$$\mathcal{E}_i^\sigma = b_{ijk}^\sigma \bar{B}_{j,k}, \quad (A46)$$

where we took into account that E_{ijmn}^σ is a symmetric tensor in indexes i and j . Thus, the effect of a mean velocity shear on the mean electromotive force, $\mathcal{E}_i^\sigma(\mathbf{r}=0) \equiv (1/2)\varepsilon_{imn} \int \Phi_{mn}^{(M,\sigma,s)} d\mathbf{k}$, is determined by

$$\mathcal{E}_i^\sigma = \varepsilon_{imn} \int \frac{\tau}{1+2\psi} \{ J_{mnpq}^\sigma \tilde{g}_{pq} + i\tau(\mathbf{k} \cdot \bar{\mathbf{B}}) \times [I_{mnpq}^\sigma \tilde{f}_{pq} + I_{mn}^{(f,\sigma)} - I_{mn}^{(h,\sigma)}] + I_{mn}^{(g,\sigma)} \} d\mathbf{k}. \quad (A44)$$

Now we use the following identities:

$$\varepsilon_{imp} \bar{K}_{jkqm} \nabla_p \bar{U}_q = \frac{1}{2} [\bar{C}_1 (2S_{ijk}^{(1)} + 2S_{ijk}^{(2)} + S_{ijk}^{(4)} + S_{ijk}^{(5)} + \bar{C}_3 (2S_{ijk}^{(6)} + S_{ijk}^{(7)})),$$

$$\varepsilon_{ijm} \bar{K}_{kmpq} \nabla_p \bar{U}_q = 2\bar{C}_1 S_{ijk}^{(1)},$$

$$\varepsilon_{ikp} \bar{K}_{jq} \nabla_p \bar{U}_q = \frac{1}{2} [\bar{A}_1 (2S_{ijk}^{(2)} + S_{ijk}^{(4)}) + \bar{A}_2 (2S_{ijk}^{(6)} + S_{ijk}^{(7)})],$$

$$\varepsilon_{imk} \bar{K}_{jmpq} \nabla_p \bar{U}_q = -2(\bar{C}_1 S_{ijk}^{(2)} + \bar{C}_3 S_{ijk}^{(6)}),$$

$$\varepsilon_{imq} \bar{K}_{mj} \nabla_k \bar{U}_q = \varepsilon_{ijm} \bar{K}_{mq} \nabla_k \bar{U}_q = \frac{1}{2} \bar{A}_1 (2S_{ijk}^{(1)} - S_{ijk}^{(5)}),$$

$$\varepsilon_{ikq} \bar{K}_{mm} \nabla_j \bar{U}_q = \frac{1}{2} (3\bar{A}_1 + \bar{A}_2) (2S_{ijk}^{(2)} - S_{ijk}^{(4)}),$$

$$\varepsilon_{imk} \bar{K}_{mq} \nabla_j \bar{U}_q = -\varepsilon_{imq} \bar{K}_{mk} \nabla_j \bar{U}_q = -\frac{1}{2} \bar{A}_1 (2S_{ijk}^{(2)} - S_{ijk}^{(4)}),$$

$$\varepsilon_{ijp} \bar{K}_{kq} \nabla_p \bar{U}_q = \frac{1}{2} \bar{A}_1 (2S_{ijk}^{(1)} + S_{ijk}^{(5)}),$$

$$\varepsilon_{ijq} \bar{K}_{mm} \nabla_k \bar{U}_q = \frac{1}{2} (3\bar{A}_1 + \bar{A}_2) (2S_{ijk}^{(1)} - S_{ijk}^{(5)}),$$

$$(\varepsilon_{ijq} \delta_{kp} + \varepsilon_{ijp} \delta_{kq}) \bar{K}_{mm} \nabla_p \bar{U}_q = 2(3\bar{A}_1 + \bar{A}_2) S_{ijk}^{(1)}, \quad (A45)$$

where

$$S_{ijk}^{(1)} = \varepsilon_{ijp} (\partial \bar{U})_{pk}, \quad S_{ijk}^{(2)} = \varepsilon_{ikp} (\partial \bar{U})_{pj},$$

$$S_{ijk}^{(3)} = \varepsilon_{jkp} (\partial \bar{U})_{pi}, \quad S_{ijk}^{(4)} = \bar{W}_k \delta_{ij}, \quad S_{ijk}^{(5)} = \bar{W}_j \delta_{ik},$$

$$S_{ijk}^{(6)} = \varepsilon_{ikp} \beta_{jq} (\partial \bar{U})_{pq}, \quad S_{ijk}^{(7)} = \bar{W}_k \beta_{ij}.$$

After the integration in Eq. (A44), we obtain

where the tensor b_{ijk}^σ is given by

$$b_{ijk}^\sigma = l_0^2 \left[\sum_{n=1}^7 D_n S_{ijk}^{(n)} \right], \quad (A47)$$

the coefficient $D_3=0$, and the other coefficients calculated for $q=5/3$ are given by

$$D_1 = \frac{1}{3} \left[A_1^{(2)} - 3A_2^{(2)} - 18C_1^{(2)} + \epsilon \left(A_1^{(2)} + A_2^{(2)} + \frac{2}{3} C_1^{(2)} \right) + \Psi_1 \left\{ A_1 + 2A_2 + \frac{22}{3} C_1 - \epsilon(2A_1 + A_2 + 6C_1) \right\} + \Psi_2 \left\{ -A_1 + \frac{1}{3} C_1 + \epsilon \left(A_1 - \frac{11}{3} C_1 \right) \right\} - (1 - \epsilon) \Psi_3 \{ C_1 \} - \Psi_0 \{ 2A_1 - 3C_1 \} \right],$$

$$D_2 = \frac{1}{3} \left[- (A_1^{(2)} + A_2^{(2)} + 4C_1^{(2)}) + \epsilon \left(-A_1^{(2)} + A_2^{(2)} + \frac{32}{3} C_1^{(2)} \right) + \Psi_1 \left\{ -A_1 + A_2 + \frac{70}{3} C_1 - 2\epsilon(A_2 + 19C_1) \right\} + \Psi_2 \left\{ A_1 - \frac{71}{3} C_1 - \epsilon \left(A_1 - \frac{79}{3} C_1 \right) \right\} + (1 - \epsilon) \left(\Psi_3 \{ -2A_1 + 7C_1 \} - \frac{16}{3} \Psi_4 \{ C_1 \} + 8\Psi_5 \{ C_1 \} \right) + \Psi_0 \left\{ 2A_1 - \frac{11}{3} C_1 \right\} \right],$$

$$D_4 = \frac{1}{6} \left[3A_1^{(2)} + A_2^{(2)} - \frac{14}{3} C_1^{(2)} + \epsilon \left(3A_1^{(2)} - A_2^{(2)} - \frac{26}{3} C_1^{(2)} \right) - \Psi_1 \left\{ A_1 + A_2 - \frac{4}{3} C_1 - 2\epsilon \left(A_1 + A_2 + \frac{2}{3} C_1 \right) \right\} + (1 - \epsilon) \times (\Psi_2 \{ A_1 + C_1 \} - \Psi_3 \{ C_1 \}) + \Psi_0 \{ C_1 \} \right],$$

$$D_5 = \frac{1}{6} \left[A_1^{(2)} + A_2^{(2)} - \frac{14}{3} C_1^{(2)} + \epsilon \left(A_1^{(2)} - A_2^{(2)} - \frac{26}{3} C_1^{(2)} \right) - \Psi_1 \left\{ A_1 - A_2 - \frac{4}{3} C_1 - 2\epsilon \left(A_1 - A_2 + \frac{2}{3} C_1 \right) \right\} + (1 - \epsilon) \times (\Psi_2 \{ A_1 + C_1 \} - \Psi_3 \{ C_1 \}) + \Psi_0 \{ C_1 \} \right],$$

$$\begin{aligned}
D_6 = & \frac{1}{3} \left[A_2^{(2)} - 4C_3^{(2)} - \epsilon \left(A_2^{(2)} - \frac{32}{3} C_3^{(2)} \right) + \Psi_1 \left\{ -3A_2 + \frac{70}{3} C_3 \right. \right. \\
& \left. \left. + 2\epsilon(A_2 - 19C_3) \right\} - \frac{1}{3} (71 - 79\epsilon) \Psi_2 \{C_3\} - (1 - \epsilon) \right. \\
& \left. \times \left(\Psi_3 \{A_2 - 7C_3\} + \frac{16}{3} \Psi_4 \{C_3\} - 8\Psi_5 \{C_3\} \right) \right. \\
& \left. + \Psi_0 \left[A_2 - \frac{11}{3} C_3 \right] \right], \\
D_7 = & \frac{1}{6} \left[A_2^{(2)} - \frac{14}{3} C_3^{(2)} + \epsilon \left(3A_2^{(2)} - \frac{26}{3} C_3^{(2)} \right) + \Psi_1 \left\{ A_2 + \frac{4}{3} \right. \right. \\
& \left. \left. \times (1 + \epsilon) C_3 \right\} + (1 - \epsilon) (\Psi_2 \{2A_2 + C_3\} - \Psi_3 \{A_2 + C_3\}) \right. \\
& \left. + \Psi_0 \{A_2 + C_3\} \right]. \tag{A48}
\end{aligned}$$

The functions $\bar{A}_k(y)$ and $\bar{C}_k(y)$ are determined by Eqs. (B6) in Appendix B, and the functions $A_k^{(2)}(y)$ and $C_k^{(2)}(y)$ are determined by Eqs. (D1) in Appendix D. The functions $\Psi_k\{X\}$ are given by

$$\begin{aligned}
\Psi_0\{X\} = & -\frac{1}{2} (1 + \epsilon) X^{(2)}(0) + (2 - \epsilon) X^{(2)}(\sqrt{2}\beta) \\
& - \frac{3}{4\pi} (1 - \epsilon) \bar{X}(2\beta^2),
\end{aligned}$$

$$\Psi_1\{X\} = -3X^{(2)}(\sqrt{2}\beta) + \frac{3}{2\pi} \bar{X}(2\beta^2),$$

$$\Psi_2\{X\} = 3X^{(2)}(\sqrt{2}\beta) - \frac{3}{4\pi} [2\bar{X}(y) + y\bar{X}'(y)]_{y=2\beta^2},$$

$$\Psi_3\{X\} = -6X^{(2)}(\sqrt{2}\beta) + \frac{3}{4\pi} [4\bar{X}(y) + y\bar{X}'(y)]_{y=2\beta^2},$$

$$\Psi_4\{X\} = 4X^{(2)}(\sqrt{2}\beta) - \frac{1}{4\pi} [8\bar{X}(y) + 4y\bar{X}'(y) + y^2\bar{X}''(y)]_{y=2\beta^2},$$

$$\begin{aligned}
\Psi_5\{X\} = & -\frac{1}{2} X^{(2)}(\sqrt{2}\beta) + \frac{1}{8\pi} [2\bar{X}(y) + y\bar{X}'(y) \\
& + 2y^2\bar{X}''(y)]_{y=2\beta^2}. \tag{A49}
\end{aligned}$$

In Eqs. (A45)–(A49) we took into account that for the shear-current dynamo, $\bar{B}_x/\bar{B}_y \sim l_0/L_B \ll 1$, where L_B is the characteristic scale of the mean magnetic field variations. The nonlinear coefficient defining the shear-current effect is determined by

$$\sigma_0(\bar{B}) = \frac{1}{2} (D_2 + 2D_4 + D_6 + 2D_7). \tag{A50}$$

Equation (A50) yields

$$\sigma_0(\bar{B}) = \phi_1 \{A_1 + A_2\} + \phi_2 \{C_1 + C_3\}, \tag{A51}$$

where

$$\begin{aligned}
\phi_1\{X\} = & \frac{1}{3} ((1 + \epsilon) X^{(2)}(\sqrt{2}\beta) + [\Psi_0 - (1 - \epsilon)(\Psi_1 - \Psi_2 + \Psi_3)] \\
& \times \{X\}), \tag{A52}
\end{aligned}$$

$$\begin{aligned}
\phi_2\{X\} = & \frac{1}{9} ((3\epsilon - 13) X^{(2)}(\sqrt{2}\beta) + [4\Psi_2 - 4\Psi_0 - 18\Psi_1 \\
& + (1 - \epsilon)(55\Psi_1 - 38\Psi_2 + 9\Psi_3 - 8\Psi_4 + 12\Psi_5)] \{X\}). \tag{A53}
\end{aligned}$$

The nonlinear dependence of the parameter $\sigma_0(\bar{B})$ determined by Eq. (A51) is shown in Fig. 3 for different values of the parameter ϵ . The asymptotic formulas for the parameter $\sigma_0(\bar{B})$ for $\bar{B} \ll \bar{B}_{\text{eq}}/4$ and $\bar{B} \gg \bar{B}_{\text{eq}}/4$ are given by Eqs. (25)–(27). For the derivation of Eq. (A51) we used the identities (D2) in Appendix D.

APPENDIX B: THE IDENTITIES USED FOR THE INTEGRATION IN k SPACE

To integrate over the angles in \mathbf{k} space we used the following identities:

$$\bar{K}_{ij} = \int \frac{k_{ij} \sin \theta}{1 + a \cos^2 \theta} d\theta d\varphi = \bar{A}_1 \delta_{ij} + \bar{A}_2 \beta_{ij}, \tag{B1}$$

$$\begin{aligned}
\bar{K}_{ijmn} = & \int \frac{k_{ijmn} \sin \theta}{1 + a \cos^2 \theta} d\theta d\varphi = \bar{C}_1 (\delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \\
& + \bar{C}_2 \beta_{ijmn} + \bar{C}_3 (\delta_{ij} \beta_{mn} + \delta_{im} \beta_{jn} + \delta_{in} \beta_{jm} + \delta_{jm} \beta_{in} \\
& + \delta_{jn} \beta_{im} + \delta_{mn} \beta_{ij}), \tag{B2}
\end{aligned}$$

$$\begin{aligned}
\bar{H}_{ijmn}(a) = & \int \frac{k_{ijmn} \sin \theta}{(1 + a \cos^2 \theta)^2} d\theta d\varphi \\
= & - \left(\frac{\partial}{\partial b} \int \frac{k_{ijmn} \sin \theta}{b + a \cos^2 \theta} d\theta d\varphi \right)_{b=1} \\
= & \bar{K}_{ijmn}(a) + a \frac{\partial}{\partial a} \bar{K}_{ijmn}(a), \tag{B3}
\end{aligned}$$

$$\begin{aligned}
\bar{G}_{ijmn}(a) = & \int \frac{k_{ijmn} \sin \theta}{(1 + a \cos^2 \theta)^3} d\theta d\varphi \\
= & -\frac{1}{2} \left(\frac{\partial}{\partial b} \int \frac{k_{ijmn} \sin \theta}{(b + a \cos^2 \theta)^2} d\theta d\varphi \right)_{b=1} \\
= & \bar{H}_{ijmn}(a) + \frac{a}{2} \frac{\partial}{\partial a} \bar{H}_{ijmn}(a), \tag{B4}
\end{aligned}$$

$$\begin{aligned}
 \bar{Q}_{ijmn}(a) &= \int \frac{k_{ijmn} \sin \theta}{(1+a \cos^2 \theta)^4} d\theta d\varphi \\
 &= -\frac{1}{3} \left(\frac{\partial}{\partial b} \int \frac{k_{ijmn} \sin \theta}{(b+a \cos^2 \theta)^3} d\theta d\varphi \right)_{b=1} \\
 &= \bar{G}_{ijmn}(a) + \frac{a}{3} \frac{\partial}{\partial a} \bar{G}_{ijmn}(a),
 \end{aligned} \tag{B5}$$

where $a = [\beta u_0 k \tau(k)/2]^2$, $\hat{\beta}_i = \beta_i / \beta$, $\beta_{ij} = \hat{\beta}_i \hat{\beta}_j$, and

$$\begin{aligned}
 \bar{A}_1 &= \frac{2\pi}{a} \left[(a+1) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 1 \right], \\
 \bar{A}_2 &= -\frac{2\pi}{a} \left[(a+3) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 3 \right], \\
 \bar{C}_1 &= \frac{\pi}{2a^2} \left[(a+1)^2 \frac{\arctan(\sqrt{a})}{\sqrt{a}} - \frac{5a}{3} - 1 \right], \\
 \bar{C}_2 &= \bar{A}_2 - 7\bar{A}_1 + 35\bar{C}_1, \\
 \bar{C}_3 &= \bar{A}_1 - 5\bar{C}_1.
 \end{aligned} \tag{B6}$$

In the case of $a \ll 1$ these functions are given by

$$\begin{aligned}
 \bar{A}_1(a) &\sim \frac{4\pi}{3} \left(1 - \frac{1}{5}a \right), & \bar{A}_2(a) &\sim -\frac{8\pi}{15}a, \\
 \bar{C}_1(a) &\sim \frac{4\pi}{15} \left(1 - \frac{1}{7}a \right), & \bar{C}_2(a) &\sim \frac{32\pi}{315}a^2, \\
 \bar{C}_3(a) &\sim -\frac{8\pi}{105}a.
 \end{aligned}$$

In the case of $a \gg 1$ these functions are given by

$$\begin{aligned}
 \bar{A}_1(a) &\sim \frac{\pi^2}{\sqrt{a}} - \frac{4\pi}{a}, & \bar{A}_2(a) &\sim -\frac{\pi^2}{\sqrt{a}} + \frac{8\pi}{a}, \\
 \bar{C}_1(a) &\sim \frac{\pi^2}{4\sqrt{a}} - \frac{4\pi}{3a}, & \bar{C}_2(a) &\sim \frac{3\pi^2}{4\sqrt{a}} - \frac{32\pi}{3a}, \\
 \bar{C}_3(a) &\sim -\frac{\pi^2}{4\sqrt{a}} + \frac{8\pi}{3a}.
 \end{aligned}$$

APPENDIX C: THE FUNCTIONS $A_n^{(1)}(\beta)$ AND $C_n^{(1)}(\beta)$

The functions $A_n^{(1)}(\beta)$ are defined as

$$A_n^{(1)}(\beta) = \frac{3\beta^4}{\pi} \int_{\beta}^{\infty} \frac{\bar{A}_n(X^2)}{X^5} dX,$$

and similarly for $C_n^{(1)}(\beta)$, where

$$X^2 = \beta^2 (k/k_0)^{2/3} = a = [\beta u_0 k \tau(k)/2]^2,$$

and we took into account that the inertial range of the turbulence exists in the scales: $l_d \leq r \leq l_0$. Here the maximum scale

of the turbulence $l_0 \ll L_B$, and $l_d = l_0 \text{Re}^{3/4}$ is the viscous scale of turbulence, and L_B is the characteristic scale of variations of the nonuniform mean magnetic field. For very large Reynolds numbers $k_d = l_d^{-1}$ is very large and the turbulent hydrodynamic and magnetic energies are very small in the viscous dissipative range of the turbulence $0 \leq r \leq l_d$. Thus we integrated in \bar{A}_n over k from $k_0 = l_0^{-1}$ to ∞ . We also used the identity

$$\int_0^1 \bar{A}_n(a(\bar{\tau})) \bar{\tau} d\bar{\tau} = \frac{2\pi}{3} A_n^{(1)}(\beta),$$

and similarly for $C_n^{(1)}(\beta)$. The functions $A_n^{(1)}(\beta)$ and $C_n^{(1)}(\beta)$ are given by

$$A_1^{(1)}(\beta) = \frac{6}{5} \left[\frac{\arctan \beta}{\beta} \left(1 + \frac{5}{7\beta^2} \right) + \frac{1}{14} L(\beta) - \frac{5}{7\beta^2} \right],$$

$$A_2^{(1)}(\beta) = -\frac{6}{5} \left[\frac{\arctan \beta}{\beta} \left(1 + \frac{15}{7\beta^2} \right) - \frac{2}{7} L(\beta) - \frac{15}{7\beta^2} \right],$$

$$C_1^{(1)}(\beta) = \frac{3}{10} \left[\frac{\arctan \beta}{\beta} \left(1 + \frac{10}{7\beta^2} + \frac{5}{9\beta^4} \right) + \frac{2}{63} L(\beta) - \frac{235}{189\beta^2} - \frac{5}{9\beta^4} \right],$$

$$C_2^{(1)}(\beta) = A_2^{(1)}(\beta) - 7A_1^{(1)}(\beta) + 35C_1^{(1)}(\beta),$$

$$C_3^{(1)}(\beta) = A_1^{(1)}(\beta) - 5C_1^{(1)}(\beta), \tag{C1}$$

where $L(\beta) = 1 - 2\beta^2 + 2\beta^4 \ln(1 + \beta^{-2})$. For $\beta \ll 1$ these functions are given by

$$A_1^{(1)}(\beta) \sim 1 - \frac{2}{5}\beta^2, \quad A_2^{(1)}(\beta) \sim -\frac{4}{5}\beta^2,$$

$$C_1^{(1)}(\beta) \sim \frac{1}{5} \left(1 - \frac{2}{7}\beta^2 \right), \quad C_2^{(1)}(\beta) \sim -\frac{32}{105}\beta^4 \ln \beta,$$

$$C_3^{(1)}(\beta) \sim -\frac{4}{35}\beta^2,$$

and for $\beta \gg 1$ they are given by

$$A_1^{(1)}(\beta) \sim \frac{3\pi}{5\beta} - \frac{2}{\beta^2}, \quad A_2^{(1)}(\beta) \sim -\frac{3\pi}{5\beta} + \frac{4}{\beta^2},$$

$$C_1^{(1)}(\beta) \sim \frac{3\pi}{20\beta} - \frac{2}{3\beta^2}, \quad C_2^{(1)}(\beta) \sim \frac{9\pi}{20\beta},$$

$$C_3^{(1)}(\beta) \sim -\frac{3\pi}{20\beta} + \frac{4}{3\beta^2}.$$

Here we used that for $\beta \ll 1$ the function $L(\beta) \sim 1 - 2\beta^2 - 4\beta^4 \ln \beta$, and for $\beta \gg 1$ the function $L(\beta) \sim 2/3\beta^2$. We also use the identity

$$\int_0^1 \bar{H}_{ijmn}(a(\bar{\tau})) \bar{\tau}^4 \left(\frac{k}{k_0}\right)^2 d\bar{\tau} = 2\pi K_{ijmn}^{(1)}(\beta) - \bar{K}_{ijmn}(\beta^2).$$

APPENDIX D: THE FUNCTIONS $A_\alpha^{(2)}(\beta)$ AND $C_\alpha^{(2)}(\beta)$

The functions $A_n^{(2)}(\beta)$ are defined as

$$A_n^{(2)}(\beta) = \frac{3\beta^6}{\pi} \int_\beta^\infty \frac{\bar{A}_n(X^2)}{X^7} dX,$$

and similarly for $C_n^{(2)}(\beta)$. We used the identity

$$\int_0^1 \bar{A}_n(a(\bar{\tau})) \bar{\tau}^2 d\bar{\tau} = \frac{2\pi}{3} A_n^{(2)}(\beta),$$

and similarly for $C_n^{(2)}(\beta)$. The functions $A_\alpha^{(2)}(\beta)$ and $C_\alpha^{(2)}(\beta)$ are given by

$$\begin{aligned} A_1^{(2)}(\beta) &= F(1; -1; 0), \\ A_2^{(2)}(\beta) &= F(-1; 3; 0), \\ C_1^{(2)}(\beta) &= (1/4)F(1; -2; 1), \\ C_2^{(2)}(\beta) &= (1/4)F(3; -30; 35), \\ C_3^{(2)}(\beta) &= (1/4)F(-1; 6; -5), \end{aligned} \tag{D1}$$

where

$$F(\alpha; \sigma; \gamma) = \pi[\alpha J_0^{(2)}(\beta) + \sigma J_2^{(2)}(\beta) + \gamma J_4^{(2)}(\beta)],$$

$$J_0^{(2)}(\beta) = \frac{1}{7\pi} \left(1 + 6 \frac{\arctan \beta}{\beta} - \frac{3\beta^2}{2} L(\beta) \right),$$

$$J_2^{(2)}(\beta) = \frac{7}{9} J_0^{(2)}(\beta) + \tilde{L}(\beta),$$

$$J_4^{(2)}(\beta) = \frac{9}{11} \left(J_2^{(2)}(\beta) - \frac{1}{\beta^2} \tilde{L}(\beta) - \frac{4}{9\pi\beta^2} \right),$$

$$\tilde{L}(\beta) = \frac{2}{3\pi\beta^2} \left(1 - \frac{\arctan \beta}{\beta} (1 + \beta^2) \right).$$

For $\beta \ll 1$ the functions $J_\alpha^{(2)}(\beta)$ are given by

$$J_0^{(2)}(\beta) \sim \frac{1}{\pi} \left(1 - \frac{1}{2} \beta^2 \right),$$

$$J_2^{(2)}(\beta) \sim \frac{1}{3\pi} \left(1 - \frac{9}{10} \beta^2 \right),$$

$$J_4^{(2)}(\beta) \sim \frac{1}{5\pi} \left(1 - \frac{15}{14} \beta^2 \right),$$

and for $\beta \gg 1$ they are given by

$$J_0^{(2)}(\beta) \sim \frac{3}{7\beta} - \frac{3}{4\pi\beta^2},$$

$$J_2^{(2)}(\beta) \sim \frac{3}{4\pi\beta^2},$$

$$J_4^{(2)}(\beta) \sim \frac{1}{4\pi\beta^2}.$$

For $\beta \ll 1$ the functions $A_\alpha^{(2)}(\beta)$ and $C_\alpha^{(2)}(\beta)$ are given by

$$A_1^{(2)}(\beta) \sim \frac{2}{3} \left(1 - \frac{3}{10} \beta^2 \right), \quad A_2^{(2)}(\beta) \sim -\frac{2}{5} \beta^2,$$

$$C_1^{(2)}(\beta) \sim \frac{2}{15} \left(1 - \frac{3}{14} \beta^2 \right), \quad C_2^{(2)}(\beta) \sim O(\beta^4),$$

$$C_3^{(2)}(\beta) \sim -\frac{2}{35} \beta^2,$$

and for $\beta \gg 1$ they are given by

$$A_1^{(2)}(\beta) \sim \frac{3\pi}{7\beta} - \frac{3}{2\beta^2}, \quad A_2^{(2)}(\beta) \sim \frac{3\pi}{7\beta} + \frac{3}{\beta^2},$$

$$C_1^{(2)}(\beta) \sim \frac{3\pi}{28\beta} - \frac{1}{2\beta^2}, \quad C_2^{(2)}(\beta) \sim \frac{9\pi}{28\beta} - \frac{4}{\beta^2},$$

$$C_3^{(2)}(\beta) \sim \frac{3\pi}{28\beta} + \frac{1}{\beta^2}.$$

We also used the following identities:

$$\Psi_1\{K_{ijmn}\} = K_{ijmn}^{(2)}(\sqrt{2}\beta) - H_{ijmn}^{(2)}(\sqrt{2}\beta),$$

$$\Psi_2\{K_{ijmn}\} = K_{ijmn}^{(2)}(\sqrt{2}\beta) - 2H_{ijmn}^{(2)}(\sqrt{2}\beta) + G_{ijmn}^{(2)}(\sqrt{2}\beta),$$

$$\Psi_3\{K_{ijmn}\} = H_{ijmn}^{(2)}(\sqrt{2}\beta) - G_{ijmn}^{(2)}(\sqrt{2}\beta),$$

$$\Psi_4\{K_{ijmn}\} = H_{ijmn}^{(2)}(\sqrt{2}\beta) - 2G_{ijmn}^{(2)}(\sqrt{2}\beta) + Q_{ijmn}^{(2)}(\sqrt{2}\beta),$$

$$\begin{aligned} \Psi_5\{K_{ijmn}\} &= \frac{1}{2} [K_{ijmn}^{(2)}(\sqrt{2}\beta) - 3H_{ijmn}^{(2)}(\sqrt{2}\beta) + 3G_{ijmn}^{(2)}(\sqrt{2}\beta) \\ &\quad - Q_{ijmn}^{(2)}(\sqrt{2}\beta)], \end{aligned} \tag{D2}$$

where

$$H_{ijmn}^{(2)}(\sqrt{2}\beta) = 4K_{ijmn}^{(2)}(\sqrt{2}\beta) - \frac{3}{2\pi} \bar{K}_{ijmn}(2\beta^2),$$

$$G_{ijmn}^{(2)}(\sqrt{2}\beta) = \frac{5}{2} H_{ijmn}^{(2)}(\sqrt{2}\beta) - \frac{3}{4\pi} \bar{H}_{ijmn}(2\beta^2),$$

$$Q_{ijmn}^{(2)}(\sqrt{2}\beta) = 2G_{ijmn}^{(2)}(\sqrt{2}\beta) - \frac{1}{2\pi} \bar{G}_{ijmn}(2\beta^2).$$

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